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Brief Paper

Minimizing the effect of out-of-bandwidth dynamics in the models of reverberant systems that arise in modal analysis: implications on spatial $\mathscr{H}_\infty$ Control☆

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Abstract

The modal approach to modeling of structures and acoustic systems results in infinite-dimensional models. For control design purposes, these models are simplified by removing higher frequency modes which lie out of the bandwidth of interest. Truncation can considerably perturb zeros of the truncated model. This paper suggests a method of minimizing the effect of removed higher-order modes on the spatial low-frequency dynamics of the truncated model by adding a spatial zero-frequency term to the low-order model of the system. The paper also studies implications of this approach on spatial $\mathscr{H}_\infty$ control of reverberant systems. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Flexible structures; Model reduction; Spatial norms; Partial differential equations

1. Introduction

The modal analysis approach has been extensively used throughout the literature to model dynamics of distributed parameter systems. Such systems include, but are not limited to, flexible beams and plates (Meirovitch, 1986), slewing beams (Fraser & Daniel, 1991; Book & Hastings, 1987), piezoelectric laminate beams (Alberts & Colvin, 1991) and acoustic ducts (Hong et al., 1996). These systems share the property that dynamics of each one of them is described by a particular partial differential equation. In this modeling technique the solutions of these partial differential equations are assumed to consist of an infinite number of terms. Moreover, these terms are chosen to be orthogonal. Hence, modeling of a system using this approach can result in an infinite-dimensional model.

In control design problems, one is often only interested in designing a controller for a particular frequency range. In these situations, it is a common practice to remove the modes which correspond to frequencies that lie out of the bandwidth of interest and only keep the modes which directly contribute to the low-frequency dynamics of the system. This model is then used to design a controller. If such a controller is implemented on the system, say in the laboratory, the closed-loop performance of the system can be considerably different from the theoretical predictions. This is mainly due to the fact that although the poles of the truncated system are at the correct frequencies, the zeros can be far away from where they should be. Therefore, it is natural to expect that a controller designed for the truncated system, may not perform well when implemented on the real system since the closed loop performance of the system is largely dictated by the open-loop zeros.

To this end, we point out that there are alternative methods to the modal approach for modeling of distributed parameter systems. As an example, one can point to the recent works of Pota and Alberts in modeling of such systems using symbolic computations (Pota & Alberts, 1995, 1997; Alberts, DuBois & Pota, 1995). However, the modal models have the interesting property that they describe spatial and temporal behavior of the system. Such models can then be used in designing spatial

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controllers as noted in Moheimani, Pota and Petersen (1997a,b, 1999, 1998a,b)

The concept of *Spatial Control* is concerned with using the spatial information embedded in the dynamical models of structures. These models are derived using the modal analysis approach. This information is then used in the controller design phase such that a level of performance is guaranteed for the entire structure. The spatial controller design methodologies suggested in Moheimani et al. (1997a,b, 1998a, 1999), result in controllers that are of the same dimension as that of the dynamical system. Therefore, it is natural to reduce the order of model used in the controller design phase. Moheimani et al. (1999) attempts to address this issue through a spatial balanced model reduction approach. The procedure suggested in this article, however, is different to Moheimani et al. (1999) in that the spatial DC content of the truncated model of the system is corrected in an optimal way. A similar technique was used in Moheimani (1999) to correct point-wise truncated models of reverberant systems when spatial characteristics of the system is not of interest.

In this paper, we show that the spatial effect of the truncated modes on the low-frequency dynamics of the system can be captured by adding a spatial zero-frequency term to the truncated model of the system. Moreover, we explain how this term can be found such that the spatial $\mathscr{H}_2$ norm of the error system is minimized. Furthermore, we will show that by embedding these spatial zero frequency terms in the truncated models of reverberant systems, the singularity problem which naturally occurs in spatial $\mathscr{H}_\infty$ control of these systems is avoided.

The remainder of the paper continues as follows. In Section 2 we briefly review the modal analysis approach to modeling of reverberant systems. We will mention a number of systems whose dynamical models can be obtained by employing the modal analysis technique. We will also show that these models have the distinct property that they describe the spatial as well as temporal behavior of the system. In Section 3 we address the problem of minimizing the effect of out of bandwidth modes on the truncated models of a particular class of reverberant systems by adding a spatial zero-frequency term to the truncated model. To do this, we introduce the notion of Spatial $\mathscr{H}_2$ norm, which is a natural extension of the $\mathscr{H}_2$ norm to the particular class of systems that are studied in this paper. In Section 4, we extend the results of Section 3 to the case of systems with multiple inputs. Section 5 studies the implications of this modeling technique on the spatial $\mathscr{H}_\infty$ control of reverberant plants. Finally, Section 6 contains an illustrative example.

2. Modal analysis

In this section, we review the mathematical basis upon which a large number of reverberant systems are modeled. We consider a partial differential equation described by

$$\mathscr{L}\{y(t,r)\} + \mathscr{M}\left\{\frac{\partial^2 y(t,r)}{\partial t^2}\right\} = f(t,r). \tag{2.1}$$

Here, r is defined over a domain $\mathscr{R}$, $\mathscr{L}$ is a linear homogeneous differential operator of order $2p$, $\mathscr{M}$ is a linear homogeneous differential operator of order $2q$, $q \le p$ and $f(t,r)$ is the system input, which could be spatially distributed over $\mathscr{R}$. Corresponding to this partial differential equation are the following boundary conditions:

$$\mathscr{B}_i\{y(t,r)\} = 0, \quad i = 1,2,\dots,p,\ r \in \mathscr{R}. \tag{2.2}$$

We notice that (2.1) and (2.2) describe *spatial* and *temporal* behavior of y. It is our intention to explain how a model of y can be derived that captures the spatial and temporal characteristics of (2.1) and (2.2). The modal analysis is concerned with assuming a solution for (2.1) in the form

$$y(t,r) = \sum_{i=1}^{\infty} \phi_i(r) q_i(t). \tag{2.3}$$

Here $\phi_i(\cdot)$ are the eigenfunctions that are obtained by solving the eigenvalue problem associated with (2.1). That is, $\mathscr{L}\{Y(r)\} = \omega^2 \mathscr{M}\{Y(r)\}$ and its associated boundary conditions, $\mathscr{B}_i = 0, i = 1,2,\dots$. In the modal analysis literature, ϕ_i's are often referred to as mode shapes. The mode-shapes are chosen to satisfy the following orthogonality conditions:

$$\int_{\mathscr{R}} \phi_i(r) \mathscr{L}\{\phi_j(r)\}\, \mathrm{d}r = \delta_{ij}\omega_i^2, \tag{2.4}$$

$$\int_{\mathscr{R}} \phi_i(r) \mathscr{M}\{\phi_j(r)\}\, \mathrm{d}r = \delta_{ij}, \tag{2.5}$$

where δ_{ij} is the Kronecker delta function, i.e., $\delta_{ij} = 1$ for $i = j$, and zero otherwise.

Substituting (2.3) in (2.1), we obtain

$$\mathscr{L}\left\{\sum_{i=1}^{\infty} \phi_i(r) q_i(t)\right\} + \mathscr{M}\left\{\frac{\partial^2}{\partial t^2}\sum_{i=1}^{\infty} \phi_i(r) q_i(t)\right\} = f(t,r). \tag{2.6}$$

Multiplying both sides of (2.6) by $\phi_j(r)$, integrating over the domain $\mathscr{R}$ and taking advantage of the orthogonality conditions (2.4) and (2.5), we obtain an infinite number of decoupled second-order ordinary differential equations: $\ddot{q}_i(t) + \omega_i^2 q_i(t) = Q_i(t)$, $i = 1,2,\dots,$ where $Q_i(t) = \int_{\mathscr{R}} \phi_i(r) f(t,r)\, \mathrm{d}r$. In many cases, $Q_i(t)$ can be written as $Q_i(t) = F_i u(t)$ where $u(t)$ is the input of the system. That is, $f(t,r)$ can be decomposed into its spatial and temporal components. Taking the Laplace transform of the second-order differential equations we obtain the input–output equation of the system in terms of a transfer function: $G(s,r) = \sum_{i=1}^{\infty} \phi_i(r) F_i/(s^2 + \omega_i^2)$.

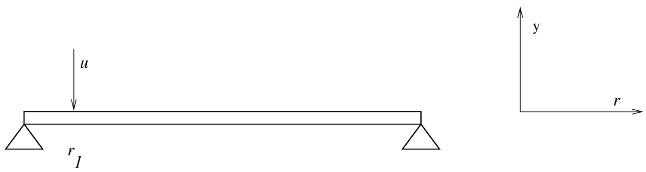

Fig. 1. A simply supported flexible beam.

As explained earlier, there are a large number of systems whose models can be obtained using the above technique. Now, we mention one of these systems that satisfy (2.1) and (2.2), and hence, modal analysis can be employed to derive a model for this system. The system is a simply supported beam that is subject to a point force.

Consider a simply supported beam as depicted in Fig. 1. Here, $y(t,r)$ denotes the elastic deformation of the beam as measured from the rest position. The elastic deflection $y(t,r)$ is governed by the classical Bernoulli–Euler beam equation (Meirovitch, 1986):

$$\frac{\partial^2}{\partial r^2}\left[EI\frac{\partial^2 y(t,r)}{\partial r^2}\right] + \rho A\frac{\partial^2 y(t,r)}{\partial t^2} = u(t)\delta(r-r_1), \tag{2.7}$$

where E, I, A, $u(t)\delta(r-r_1)$ and ρ represent, respectively, the Young's modulus, moment of inertia, cross-section area, external force applied at r_1, and the linear mass density of the beam.

Pinned–pinned beam boundary conditions are

$$y(t,0)=0, \qquad y(t,L)=0, \qquad EI\frac{\partial^2 y(t,r)}{\partial r^2}\bigg|_{r=0} = 0,$$

$$EI\frac{\partial^2 y(t,r)}{\partial r^2}\bigg|_{r=L} = 0. \tag{2.8}$$

The first two boundary conditions state that there are no movements at the two ends of the beam and the last two conditions state that the beam curvatures at both ends are zero.

Comparing (2.7) and (2.8) with (2.1) and (2.2), we notice that $\mathscr{L} = (\mathrm{d}^2/\mathrm{d}r^2)(EI(\mathrm{d}^2/\mathrm{d}r^2))$; $\mathscr{M} = \rho A$, $\mathscr{B}_1 = 1$; $\mathscr{B}_2 = EI(\mathrm{d}^2/\mathrm{d}r^2)$ and $f(t,r) = u(t)\delta(r-r_1)$.

Assuming a solution of form (2.3) and following the procedure that was explained earlier, we can find a transfer function for this system. The eigenfunctions are chosen to be orthogonal according to the condition,

$$\int_0^L \phi_i(r)\phi_j(r)\rho A\,\mathrm{d}r = \delta_{ij}. \tag{2.9}$$

The transfer function between applied force $U(s)$ and the elastic deflection of the beam $\hat{y}(s,r)$ is given by (Krishnan and Vidyasagar, 1987):

$$\frac{\hat{y}(s,r)}{U(s)} = \sum_{i=1}^{\infty}\frac{\phi_i(r_1)\phi_i(r)}{(s^2+\omega_i^2)}. \tag{2.10}$$

For the pinned–pinned beam system in Fig. 1 the mode functions are given by (Meirovitch, 1986)

$$\phi_i(r) = \sqrt{\frac{2}{\rho A L}}\sin\left(\frac{\mathrm{i}\pi r}{L}\right) \tag{2.11}$$

and the corresponding natural frequencies are $\omega_i = (\mathrm{i}\pi/L)^2\sqrt{EI/\rho A}$.

3. Statement of the problem

In general, the model of a reverberant system, such as a flexible structure consists of an infinite number of terms. Let the transfer function $G(s)$ be as follows:

$$G(s,r) = \sum_{i=1}^{\infty}\frac{F_i\phi_i(r)}{s^2+\omega_i^2}, \tag{3.1}$$

where r belongs to a known set, i.e., $r\in\mathscr{R}$. For a beam, this would be an interval, i.e. $\mathscr{R} = [0, L]$, where L is the length of the beam. For a plate, $\mathscr{R}$ would be a two-dimensional set, $\mathscr{R} = \{(r_1, r_2)\in\mathbf{R}\colon r_1\in[0, L_1], r_2\in[0, L_2]\}$, where L_1 and L_2 represent the width and the length of the plate. The orthogonality condition corresponding to (3.1) is as follows:

$$\int_{\mathscr{R}} \phi_i(r)\phi_j(r)\,\mathrm{d}r = \Phi_i^2\delta_{ij}. \tag{3.2}$$

This orthogonality condition is a particular case of (2.5), which holds for a large number of systems such as beams and plates with uniform mass distribution, acoustic enclosures with uniform cross sections, uniform strings, etc.

This is an infinite-dimensional transfer function due to the existence of an infinite number of modes. Moreover, G describes the spatial as well as the temporal behavior of the structure. In a typical control design scenario, the designer is often interested only in a particular bandwidth. Therefore, an approximate model of the system is needed that best represents the dynamics of the system in the prescribed frequency range. Hence, a lower-order dynamical model is needed. A natural choice in this case is to simply ignore the modes which correspond to the frequencies that lie outside of the bandwidth of interest. For instance, if ω_{N+1} is larger than the highest frequency of interest, one may choose to approximate $G(s,r)$ by

$$G_N(s,r) = \sum_{i=1}^{N}\frac{F_i\phi(r)}{s^2+\omega_i^2}. \tag{3.3}$$

This seems to be the mainstream approach in simplifying the dynamics of reverberant systems as explained in Clark (1997). A drawback of this approximation is that the truncated higher-order modes may contribute to the low-frequency dynamics, mainly in the form of distorting zero locations. Furthermore, these removed modes could significantly distort the spatial characteristics of the low-order model. Therefore, an approximate low-order model is needed that best captures the effect of truncated modes on the temporal and spatial dynamics of the structure.

Reference Clark (1997) suggests a way of dealing with this problem when the spatial behavior of the structure is not of importance i.e., when the observation point is set at a particular position along the structure. In the aeroelasticity literature this approach is referred to as the mode acceleration method (see p. 350 of Bisplinghoff & Ashley, 1962). The idea is that given r, e.g. $r = r_2$, one can allow for a constant feed-through term in $G_N(s, r_2)$ to account for the compliance of the truncated higher-order modes of $G(s, r_2)$. That is, to approximate $G(s, r_2)$ by

$$\hat{G}(s, r_2) = G_N(s, r_2) + K, \tag{3.4}$$

where $K = \sum_{i=N+1}^{\infty} F_i \phi_i(r_2)/\omega_i^2$. The logic behind this choice of K is that at lower frequencies one can ignore the effect of dynamical responses of higher-order modes since they are much smaller than the force responses at those frequencies. Although an approximation, Clark (1997) shows that K is a good representation of the effect of higher-order modes on $G_N(s, r_2)$.

Keeping this in mind, it is natural to extend this approach to a more general transfer function such as (3.1). In this case, (3.4) can be modified to $\hat{G}(s, r) = G_N(s, r) + K(r)$ where

$$K(r) = \sum_{i=N+1}^{\infty} \frac{F_i}{\omega_i^2} \phi_i(r). \tag{3.5}$$

Adding this term to the truncated model can significantly correct the in-bandwidth spatial dynamics of the system. However, this model correction is by no means optimal. This paper is an attempt to find an optimal value for K such that the effect of the truncated higher-order modes on the low-frequency dynamical model of the structure is minimized and the spatial characteristic of the model is best preserved. To this end, we introduce the notion of *spatial* $\mathscr{H}_2$ norm of a single-input system (see also Moheimani & Fu, 1998).

Definition 3.1. The spatial $\mathscr{H}_2$ norm of a single input transfer function $G(s, r)$, with $r \in \mathscr{R}$ is defined to be

$$\langle\langle G(s, r) \rangle\rangle_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{\mathscr{R}} |G(j\omega, r)|^2 \, dr \, d\omega. \tag{3.6}$$

Notice that if $G(s, r)$ represents the dynamics of a beam such as the simply supported beam of Section 2, then $\langle\langle G(s, r) \rangle\rangle_2^2$ is a measure of the volume underneath the surface defined by $|G(j\omega, r)|^2$. Hence, this is a natural extension of the standard $\mathscr{H}_2$ norm of linear systems to systems of form (3.1). Similar interpretations can be made for transfer functions of plates, etc.

Our approach to enhancing the approximation made by truncating the higher-order modes of (3.1) is to add a spatially distributed zero-frequency term to the first N modes of (3.1). That is, to approximate (3.1) by

$$\hat{G}(s, r) = \sum_{i=1}^{N} \frac{F_i \phi_i(r)}{s^2 + \omega_i^2} + \sum_{i=N+1}^{\infty} k_i \phi_i(r). \tag{3.7}$$

This choice of the zero-frequency term is inspired by (3.5). However, unlike (3.5) we find the parameters k_i, $i = N+1, N+2, \ldots$ such that the following spatial cost function is minimized:

$$J = \langle\langle (G(s, r) - \hat{G}(s, r)) W(s, r) \rangle\rangle_2^2. \tag{3.8}$$

Here, G and $\hat{G}$ are defined as in (3.1) and (3.7) and $W(s, r)$ is an ideal low-pass weighting function distributed spatially over $\mathscr{R}$ with its cut-off frequency ω_c chosen to lie within the interval $\omega_c \in (\omega_N, \omega_{N+1})$. That is, $|W(j\omega, r)| = 1$ for $-\omega_c \le \omega \le \omega_c$, $r \in \mathscr{R}$ and 0 elsewhere. The reason for this choice of W will be explained soon. To this end, it should be clear that k_i's chosen to minimize (3.8) will minimize the effect of out-of-bandwidth dynamics of $G(s, r)$ on $\hat{G}(s, r)$ in a spatial $\mathscr{H}_2$ optimal sense. Therefore, guaranteeing that the reduced-order model $\hat{G}$ will best represent the frequency response of G while preserving its spatial characteristics.

Theorem 3.1. *Consider the system defined by* (3.1), *its approximation* (3.7) *and the spatial cost function* (3.8). *The parameters* k_i, $i = N+1, N+2, \ldots$ *that minimize* (3.8) *are given by*

$$k_i = \frac{F_i}{2\omega_c \omega_i} \ln\left(\frac{\omega_i + \omega_c}{\omega_i - \omega_c}\right), \quad i = N+1, N+2, \ldots\ . \tag{3.9}$$

Proof. It is straightforward to verify that (3.8) is equivalent to

$$J = \left\langle\!\!\left\langle \left(\sum_{i=N+1}^{\infty} \frac{F_i \phi(r)}{s^2 + \omega_i^2} - \sum_{i=N+1}^{\infty} k_i \phi(r) \right) W(s, r) \right\rangle\!\!\right\rangle_2^2. \tag{3.10}$$

The fact that W is chosen to be an ideal spatial low-pass filter with its cut-off frequency lower than the first out-of-bandwidth pole of G, i.e., ω_{N+1}, guarantees that (3.10) will remain finite. The cost function is then equivalent to

$$\begin{aligned} J = {} & \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \int_{\mathscr{R}} \left| \sum_{i=N+1}^{\infty} \frac{F_i \phi_i(r)}{\omega_i^2 - \omega^2} \right|^2 dr \, d\omega \\ & - 2 \times \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \int_{\mathscr{R}} \left(\sum_{i=N+1}^{\infty} \frac{F_i \phi_i(r)}{\omega_i^2 - \omega^2} \right) \\ & \times \left(\sum_{i=N+1}^{\infty} k_i \phi_i(r) \right) dr \, d\omega \\ & + \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \int_{\mathscr{R}} \left(\sum_{i=N+1}^{\infty} k_i \phi_i(r) \right)^2 dr \, d\omega, \end{aligned}$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c}\int_{\mathscr{R}}\left|\sum_{i=N+1}^{\infty}\frac{F_i\phi_i(r)}{\omega_i^2-\omega^2}\right|^2 \mathrm{d}r\,\mathrm{d}\omega$$

$$-2\left(\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c}\sum_{i=N+1}^{\infty}\frac{F_ik_i\Phi_i^2}{\omega_i^2-\omega^2}\right)\mathrm{d}\omega$$

$$+\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c}\sum_{i=N+1}^{\infty}k_i^2\Phi_i^2\,\mathrm{d}\omega,$$

where we have used the orthogonality property of the mode-shapes (3.2). The optimal set of parameters k_i, $i=N+1, N+2, \ldots$ can now be determined via setting the derivatives of J with respect to k_i equal to zero, i.e., $\partial J/\partial k_i = 0$. We then find, $k_i = (1/2\omega_c\Phi_i^2)\int_{-\omega_c}^{\omega_c} F_i\Phi_i^2/(\omega_i^2-\omega^2)\,\mathrm{d}\omega$ which is equivalent to $k_i = (F_i/2\omega_c\omega_i)\ln((\omega_i+\omega_c)/(\omega_i-\omega_c))$. □

At this point, we intend to demonstrate how the K_i obtained from Theorem 3.1 is related to the value of K_i obtained from (3.5). We know that for $x>0$, the term $\ln(x)$ can be expanded as (see p. 52 of Gradsteyn & Ryshik, 1994): $\ln(x) = 2\sum_{n=1}^{\infty}\frac{1}{2n-1}((x-1)/(x+1))^{(2n-1)}$. If we use the first term of the series to approximate $\ln((\omega_i+\omega_c)/(\omega_i-\omega_c))$ in (3.9) we obtain, $k_i = \sum_{i=N+1}^{\infty} F_i/\omega_i^2$. Hence, we recover (3.5). Therefore, the value of k_i suggested by (3.5) approximates the optimal k_i which minimizes (3.8).

To this end, we point out that the analysis given here ignores the effect of modal dampings. There are two reasons for this. First, it is a difficult exercise to determine modal dampings during the modeling phase if the modal analysis is to be used. Second, every mode of a reverberant system is very lightly damped. Therefore, it is sensible to model the structure using the modal analysis technique, then use the approximation method explained above to correct the in-bandwidth dynamics of the model, and finally measure the damping associated with each in-bandwidth mode. These dampings can then be easily embedded in the approximate model of the structure. It should be pointed out that if the modal dampings are known for a large number of modes, it is readily possible to include the dampings in the model and carry out the optimization procedure explained above with some modifications. This approach, however, is not recommended. Imagine one wishes to find a three-mode approximation of a 30-mode system. In this case, one needs to determine modal dampings for the first 30 modes. However, if the procedure discussed earlier is employed, one only needs to know the first three modal dampings.

In Section 6, we will demonstrate how effective this approximation can be in capturing the effect of higher-order modes on the frequency response of the truncated system. Moreover, we will compare the two approaches. But first, we extend the results of this section to the case of multi-input systems.

4. Extension to multi-input systems

In this section, we extend the procedure developed in Section 2, to the case of reverberant systems that are subject to more than one actuator. The transfer matrix is assumed to be of the form

$$G(s,r) = \sum_{i=1}^{\infty}\frac{\phi_i(r)}{s^2+\omega_i^2}H_i. \tag{4.1}$$

Here, $H_i = [F_1^i\ F_2^i\ \ldots\ F_m^i]$ where m is the number of actuators. Moreover, ϕ_i's are assumed to satisfy similar orthogonality conditions, i.e., (3.2). For the simply supported beam of Fig. 2 which is subject to m point forces at $r_1, \ldots, r_m$, this amounts to $F_s^i = \phi_i(r_s)/\rho A$, $s=1,2,\ldots,m$ where $\phi_i(r)$ is given by (2.11).

In parallel with Section 2, we approximate $G(s,r)$ with

$$\hat{G}(s,r) = G_N(s,r) + \Lambda(r), \tag{4.2}$$

where $\Lambda(r) = \sum_{i=N+1}^{\infty}\phi_i(r)K_i$ and $K_i = [k_1^i\ k_2^i\ \ldots\ k_m^i]$. Now, we are in a position where we can define the spatial $\mathscr{H}_2$ norm of a multi-input system such as (4.1).

Definition 4.1. Spatial $\mathscr{H}_2$ norm of a multi-input system is defined to be

$$\langle\!\langle G(s,r)\rangle\!\rangle_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\mathscr{R}} tr\{G(\mathrm{j}\omega,r)^*G(\mathrm{j}\omega,r)\}\,\mathrm{d}r\,\mathrm{d}\omega,$$

where $tr\{M\}$ represents the trace of matrix M.

Next, we consider the spatial cost function

$$J = \langle\!\langle W(s,r)(G(s,r)-\hat{G}(s,r))\rangle\!\rangle_2^2, \tag{4.3}$$

where G and $\hat{G}$ are defined as in (4.1) and (4.2). The problem here is to determine the K_i, $i=1,2,\ldots$ that minimizes the spatial cost function (4.3).

Theorem 4.1. *Consider the multi-input system* (4.1) *and its corresponding approximation* (4.2). *Then the* K_i, $i=1, 2, \ldots$ *that minimize* (4.3) *are given by*

$$K_i = \frac{1}{2\omega_c\omega_i}\ln\left(\frac{\omega_i+\omega_c}{\omega_i-\omega_c}\right)H_i, \quad i=1,2,\ldots\ . \tag{4.4}$$

Proof. It is straight-forward to show that the spatial cost function J is equivalent to

$$J = \langle\!\langle W(\mathrm{j}\omega,r)\tilde{G}(\mathrm{j}\omega,r)\rangle\!\rangle_2^2$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\mathscr{R}} tr\{\Lambda(r)'W(\mathrm{j}\omega,r)^*W(\mathrm{j}\omega,r)\Lambda(r)\}\,\mathrm{d}r\,\mathrm{d}\omega$$

$$-2\times\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\mathscr{R}}(tr\{\tilde{G}(\mathrm{j}\omega,r)^*$$

$$\times W(\mathrm{j}\omega,r)^*W(\mathrm{j}\omega,r)\Lambda(r)\})\,\mathrm{d}r\,\mathrm{d}\omega,$$

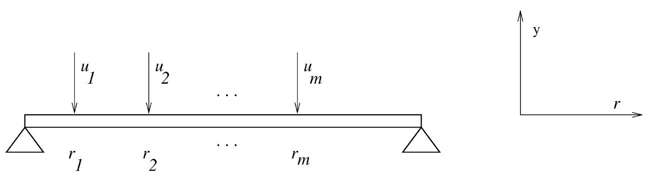

Fig. 2. A simply supported beam with m point forces.

where $\tilde{G}(s,r)=\sum_{i=N+1}^{\infty}\phi_i(r)/(s^2+\omega_i^2)H_i$. This, in turn, can be re-written as

$$J=\langle\!\langle W(\mathrm{j}\omega)\tilde{G}(\mathrm{j}\omega,r)\rangle\!\rangle_2^2+\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c}\int_{\mathscr{R}}tr\left\{\left(\sum_{i=N+1}^{\infty}\phi_i(r)K_i'\right)\times\left(\sum_{j=1}^{\infty}\phi_j(r)K_j\right)\right\}\mathrm{d}r\,\mathrm{d}\omega$$

$$-2\times\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c}\int_{\mathscr{R}}\left(tr\left\{\left(\sum_{i=N+1}^{\infty}\frac{\phi_i(r)}{\omega_i^2-\omega^2}H_i'\right)\times\left(\sum_{j=1}^{\infty}\phi_j(r)K_j\right)\right\}\right)\mathrm{d}r\,\mathrm{d}\omega.$$

Applying the orthogonality conditions to this expression for J, we obtain

$$J=\langle\!\langle W(\mathrm{j}\omega)\tilde{G}(\mathrm{j}\omega,r)\rangle\!\rangle_2^2+\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c}tr\left\{\sum_{i=N+1}^{\infty}\Phi_i^2K_i'K_i\right\}\mathrm{d}\omega-2\times\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c}tr\left\{\sum_{i=N+1}^{\infty}\frac{\Phi_i^2}{\omega_i^2-\omega^2}H_i'K_i\right\}$$

$$=\langle\!\langle W(\mathrm{j}\omega)\tilde{G}(\mathrm{j}\omega,r)\rangle\!\rangle_2^2+2\omega_c\times\frac{1}{2\pi}tr\left\{\sum_{i=N+1}^{\infty}\Phi_i^2K_i'K_i\right\}-2\times\frac{1}{2\pi}tr\left\{\sum_{i=N+1}^{\infty}\frac{\Phi_i^2}{\omega_i}\ln\left(\frac{\omega_i+\omega_c}{\omega_i-\omega_c}\right)H_i'K_i\right\}.$$

Differentiating J with respect to K_i (see p. 592 of Lewis, 1992), the optimum value of K_i is found to be: $K_i=(1/2\omega_c\omega_i)\ln((\omega_i+\omega_c)/(\omega_i-\omega_c))H_i$. □

Comparing the results of Theorems 4.1, we make the following observation.

Observation 4.1. Consider the multi-input system (4.1), and approximate each individual transfer function using the result of Theorem 3.1. Then the resulting transfer matrix is optimal in the sense of (4.3).

5. Implications on spatial $\mathscr{H}_\infty$ control of reverberant plants

The problem of spatial $\mathscr{H}_\infty$ control of reverberant plants is re-visited in this section. Consider the following dynamical system:

$$\dot{x}(t)=Ax(t)+B_1w(t)+B_2u(t),$$
$$z(t,r)=C_1(r)x(t)+D_{12}(r)u(t), \tag{5.1}$$
$$y(t)=C_2x(t)+D_{21}w(t),$$

where $x\in\mathbf{R}^n$ is the state, $w\in\mathbf{R}^p$ is the disturbance input, $u\in\mathbf{R}^m$ is the control input, z is the spatial error output and $y\in\mathbf{R}^\ell$ is the measured output. We also assume that $r\in\mathscr{R}$. The spatial $\mathscr{H}_\infty$ control problem for this system is defined as follows.

The Spatial $\mathscr{H}_\infty$ Control problem: Design a controller

$$\dot{x}_k(t)=A_kx_k(t)+B_Ky(t),$$
$$u(t)=C_kx_k(t)+D_ky(t)$$

such that the closed-loop system satisfies

$$\inf_{K(\cdot)\in U}\ \sup_{w(\cdot)\in\mathscr{L}_2[0,\infty)} J_\infty<\gamma^2, \tag{5.2}$$

where U is the set of all stabilizing controllers and

$$J_\infty=\frac{\int_0^\infty\int_{\mathscr{R}}z(t,r)'Q(r)z(t,r)\,\mathrm{d}r\,\mathrm{d}t}{\int_0^\infty w(t)'w(t)\,\mathrm{d}t}. \tag{5.3}$$

Here $Q(r)$ is a spatial weighting function that emphasizes the subset of $\mathscr{R}$ over which disturbance rejection is of importance. It should be clear that the spatial $\mathscr{H}_\infty$ controller designed to meet the performance index (5.2) and (5.3) guarantees a level of disturbance rejection over the entire $\mathscr{R}$ in an average sense while emphasizing a subset of $\mathscr{R}$ defined by $Q(r)$.

We have the following theorem:

Theorem 5.1. *Consider system* (5.1) *and its corresponding spatial $\mathscr{H}_\infty$ control problem. This problem is equivalent to a standard $\mathscr{H}_\infty$ control problem for the following system*:

$$\dot{x}(t)=Ax(t)+B_1w(t)+B_2u(t),$$
$$\tilde{z}(t)=\Pi x(t)+\Theta u(t),$$
$$y(t)=C_2x(t)+D_{21}w(t), \tag{5.4}$$

where $[\Pi\ \Theta]=\Gamma$, *and* Γ *is any matrix that satisfies*

$$\Gamma'\Gamma=\int_{\mathscr{R}}\begin{bmatrix}C(r)'\\ D_{12}(r)'\end{bmatrix}Q(r)[C(r)\ D_{12}(r)]\,\mathrm{d}r. \tag{5.5}$$

Proof. A detailed proof is not given here. However, it is pointed out that the proof is based on the observation that $\int_{\mathscr{R}}z(t,r)'Q(r)z(t,r)\,\mathrm{d}r=\tilde{z}(t)'\tilde{z}(t)$. For more in-dept coverage of spatial $\mathscr{H}_\infty$ norm and its properties, the reader is referred to Moheimani et al. (1997a, 1999). □

To make our motivation for studying this problem clearer, consider the disturbance rejection problem depicted in Fig. 3. Here, the objective is to reduce the effect of the disturbance force w over the entire beam. In this

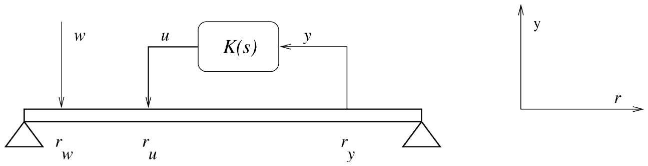

Fig. 3. Spatial $\mathscr{H}_\infty$ control of a beam.

case, the transfer functions from w and u to the elastic deflection of the beam can be modeled as

$$G_u(s,r) = \sum_{i=1}^{\infty} \frac{F_i \phi_i(r)}{s^2 + 2\zeta_i \omega_i s + \omega_i^2}, \tag{5.6}$$

$$G_w(s,r) = \sum_{i=1}^{\infty} \frac{G_i \phi_i(r)}{s^2 + 2\zeta_i \omega_i s + \omega_i^2}, \tag{5.7}$$

where F_i and G_i can be determined as explained in Section 2, i.e., $F_i = \phi_i(r_u)$ and $G_i = \phi_i(r_y)$. In this case, the problem of designing a controller to reduce the effect of the disturbance force $w(t)$ over the entire body of the beam can be cast into a spatial $\mathscr{H}_\infty$ control framework. This problem can then be reduced to an ordinary $\mathscr{H}_\infty$ control problem that, in turn, can be solved using standard software (e.g., Chiang & Safanov, 1994). However, if the transfer functions (5.6) and (5.7) are simply truncated by ignoring the out-of-bandwidth modes, the resulting $\mathscr{H}_\infty$ control problem will be singular. This is the difficulty that was addressed in Moheimani et al. (1997a). This difficulty can be avoided if the truncated higher-order terms are approximated by a zero-frequency term as explained earlier in this paper. If the first N modes are chosen to represent the in-bandwidth dynamics of the structure, G_u and G_w can be approximated by

$$G_u(s,r) = \sum_{i=1}^{N} \frac{F_i \phi_i(r)}{s^2 + 2\zeta_i \omega_i s + \omega_i^2} + \sum_{i=1}^{\infty} k_i \phi_i(r), \tag{5.8}$$

$$G_w(s,r) = \sum_{i=1}^{N} \frac{G_i \phi_i(r)}{s^2 + 2\zeta_i \omega_i s + \omega_i^2} + \sum_{i=1}^{\infty} \ell_i \phi_i(r), \tag{5.9}$$

where k_i and ℓ_i, $i = N+1, N+2, \dots$ are determined as explained in Theorem 3.1. Then the state-space model corresponding to the spatial $\mathscr{H}_\infty$ control problem can be defined as in (5.1), with

$$A = \mathrm{diag}\left(\begin{bmatrix} 0 & 1 \\ -\omega_1^2 & -2\zeta_1\omega_1 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 1 \\ -\omega_N^2 & -2\zeta_N\omega_N \end{bmatrix} \right),$$

$$B_1' = [0 \quad F_1 \quad \dots \quad 0 \quad F_N];$$

$$B_2' = [0 \quad G_1 \quad \dots \quad 0 \quad G_N],$$

$$C_1(r) = [\phi_1(r) \quad 0 \quad \dots \quad \phi_N(r) \quad 0],$$

$$C_2 = [\phi_1(r_w) \quad 0 \quad \dots \quad \phi_N(r_w) \quad 0],$$

$$D_{12}(r) = \sum_{i=1}^{\infty} k_i \phi_i(r); \qquad D_{21} = \sum_{i=1}^{\infty} \ell_i \phi_i(r_y).$$

Then, it can be shown that the spatial $\mathscr{H}_\infty$ control problem is equivalent to an ordinary $\mathscr{H}_\infty$ control problem for system (5.4) where Π is a $(2N+1)\times(2N+1)$ matrix and Θ is a $(2N+2)\times 1$ vector:

$$\Pi = diag\left(\begin{bmatrix} \Phi_1 & 0 \\ 0 & 0 \end{bmatrix}, \dots, \begin{bmatrix} \Phi_N & 0 \\ 0 & 0 \end{bmatrix} \right),$$

$$\Theta = \begin{bmatrix} 0_{2N\times 1} \\ (\sum_{i=N+1}^{\infty} k_i \Phi_i^2)^{1/2} \end{bmatrix}.$$

6. An illustrative example

In this section, we apply the model correction method which was developed in Section 4 to a simply supported beam model. The beam is shown in Fig. 1. The parameters of the beam are: $L = \text{Beam Length} = 1.3$ m, $r_1 = 0.05$ m, $\rho A = 0.6265$ kg/m, $EI = 5.329$ N m^2. Moreover, in our simulations we allow for a damping ratio of 0.3% for all the modes.

In this example, we are only interested in the first two modes of the beam. Fig. 4 shows the spatial frequency response of the beam up to a frequency of 100 rad/s. The model is obtained using the first 30 pinned–pinned modes. Fig. 5 shows the spatial frequency response of the beam using only the first two modes. To have a clear picture of the spatial error caused by truncating the higher frequency modes, we plot the frequency response

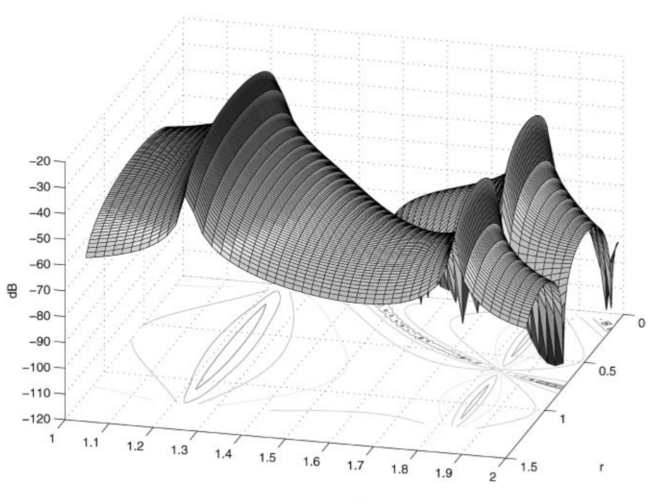

Fig. 4. Spatial frequency response of the beam based on a 30-mode model.

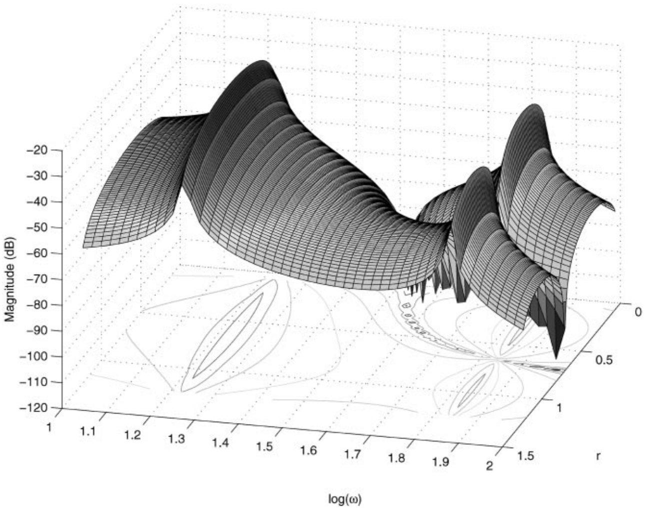

Fig. 5. Spatial frequency response of the beam based on a two-mode model.

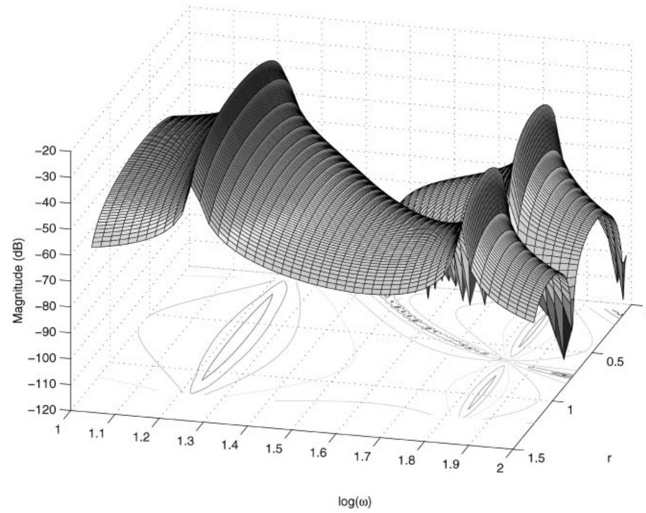

Fig. 7. Spatial frequency response of the beam based on two-mode model and a correction term.

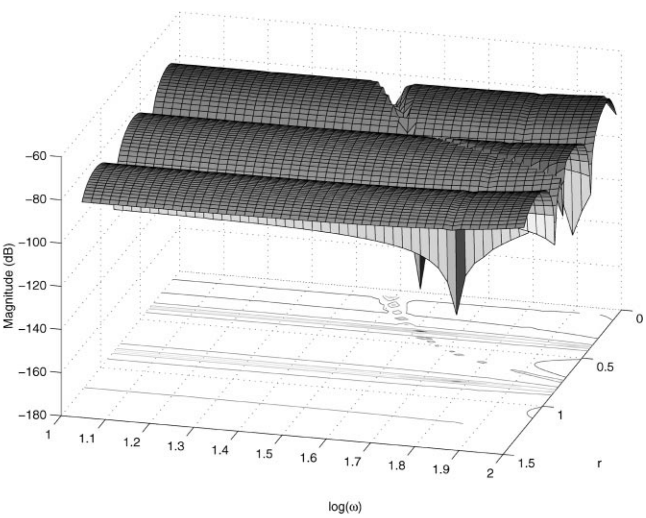

Fig. 6. Spatial frequency response of the error system (30-mode model and two-mode model).

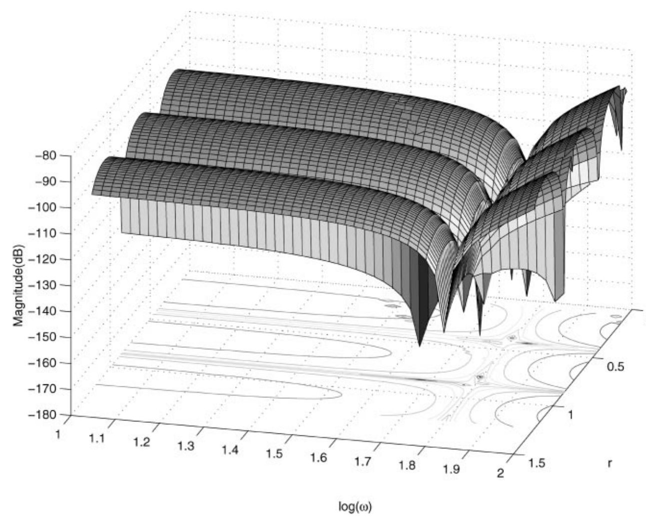

Fig. 8. Spatial frequency response of the error system (30-mode model and two-mode model plus the correction term).

of the error system in Fig. 6. Now, we approximate the truncated higher-order modes by a spatial zero-frequency term as explained in the Section 4. The spatial frequency response of this new system is plotted in Fig. 7. Finally, we plot the spatial frequency response of the error system, i.e., the 30-mode model and the two-mode model plus the correction term, in Fig. 8. It can be observed that the approximation technique suggested in this paper is a much better option than simply truncating the model as is clear from Figs. 6 and 8. Moreover, a direct consequence of this reduced error is that the in-bandwidth zeros of the corrected model are considerably closer to the real in-bandwidth zeros of the system.

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