Synthesis of Minimax Optimal Controllers for Uncertain Time-Delay Systems with Structured Uncertainty *

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Abstract

This paper is concerned with the design of robust state feedback controllers for a class of uncertain time-delay systems. The uncertainty is assumed to satisfy a certain integral quadratic constraint. The controller proposed is a minimax optimal controller in the sense that it minimizes the maximum value of a corresponding linear quadratic cost function over all admissible uncertainties. The controller leads to an absolutely stable closed loop uncertain system and is constructed by solving a finite dimensional parameter dependent algebraic Riccati equation.

Key Words: Time-delay systems, uncertain systems, absolute stability, robust performance, Riccati equations.

1 INTRODUCTION

Delays often occur in the transmission of material or information between different parts of a system. Communication systems, transmission systems, chemical processing systems, metallurgical processing systems, environmental systems and power systems are examples of time-delay systems. Considerable research has been done on various aspects of time-delay systems during past thirty years; e.g., see (Malek-Zavarei and Jamshidi 1987) and references therein.In this paper, we extend the linear quadratic regulator to the case in which the underlying system contains time-delays and is uncertain with structured uncertainty and time delays.

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The linear quadratic optimal control approach to the design of state feedback control laws for time-delay systems has been subject of intensive research over past twenty years (see for example (Malek-Zavarei and Jamshidi 1987, Alekal *et al.* 1971, Delfour *et al.* 1975, Ichikawa 1982, Lee 1980, Uchida *et al.* 1988) and references therein). An advantage of the linear quadratic optimal state feedback control approach is that the closed loop system has desirable sensitivity and robustness properties. Indeed, in (Lee and Levy 1982) and (Uchida and Shimemura 1986) it is shown that such optimal regulators satisfy a circle condition similar to the classical LQ controllers; e.g., see Section 5.4 of (Anderson and Moore 1990). That is, they guarantee classical measures of robustness, gain margin and phase margin. However, more complex forms of uncertainty have not yet been addressed in the literature concerning the LQ optimal controller design for uncertain time-delay systems. Furthermore, a serious drawback of the LQ optimal control design method for time-delay systems is that it requires solutions to infinite dimensional Riccati equations.

The problem of designing robust controllers to stabilize uncertain systems without time-delays has been subject of much research; e.g., see (Dorato *et al.* 1993) and references therein. There are many different classes of uncertainty considered in the robust control literature and for each particular class, several approaches have been proposed. An important class of such systems is the so called norm bounded uncertain systems. A widely accepted approach to designing stabilizing controllers for such systems is the quadratic stabilizability approach (Barmish 1983). This involves finding a fixed quadratic Lyapunov function that guarantees asymptotic stability of the uncertain system for all admissible values of the uncertainty. A Riccati equation approach to the quadratic stabilization of norm bounded uncertain systems is discussed in (Petersen 1987). Another important class of uncertain systems is the class of systems which satisfy a certain integral quadratic constraint. These systems are subject of currently ongoing research (Savkin and Petersen 1994c, Savkin and Petersen 1995, Savkin and Petersen 1994a, Megretsky 1994, Megretsky and Rantzer 1995). This uncertainty description leads to a very rich class of uncertainties allowing for nonlinear, time-varying, dynamic uncertainty. Indeed, the norm bounded uncertainty discussed above also fits into the integral quadratic constraint framework.

In (Petersen and McFarlane 1994), the linear quadratic regulator is extended to the case of norm-bounded uncertain systems. The controller is constructed by solving a certain game type Riccati equation. The robust controller of (Petersen and McFarlane 1994) guarantees robust stability as well as a level of robust performance measured with a linear quadratic cost function. The problem of designing a robust minimax optimal controller for an uncertain system with structured uncertainty satisfying an integral quadratic constraint is considered in (Savkin and Petersen 1995). The robust controller of (Savkin and Petersen 1995) is obtained by solving a parameter dependent Riccati equation.

The problem of designing robust controllers for uncertain time-delay systems has been subject of some recent research in (Moheimani and Petersen 1995, Trinh and Aldeen 1994, Shen *et al.* 1991). A natural approach to tackle such problems is to extend techniques proposed in the robust control literature to uncertain time-delay systems. The problem of designing guaranteed cost state feedback controllers for a class of norm bounded uncertain systems is considered in (Moheimani and Petersen 1995). This is an extension of the problem considered in (Petersen and McFarlane 1994) to the case of uncertain time-delay systems with a state time-delay. The controller stabilizes the uncertain system and guarantees an upper bound on a LQ cost function. The solution of the problem is given in terms of a finite dimensional parameter dependent Riccati equation. The upper bound on the LQ bound can then be minimized by searching over the set of parameters for which the corresponding Riccati equation has a positive definite stabilizing solution.

In this paper we extend the linear quadratic regulator to the case of uncertain time-delay systems with structured uncertainty and delays. The class of systems considered here is richer than that of (Moheimani and Petersen 1995) since we allow for structured uncertainty and time delays. Also, the uncertainty is assumed to satisfy a certain *Integral Quadratic Constraint*. This is a rich class of uncertain systems allowing for nonlinear, dynamical uncertainty; e.g., see (Savkin and Petersen 1995, Savkin and Petersen 1994c). Hence, the class of uncertain time-delay systems considered in this paper is much richer than those considered in (Moheimani and Petersen 1995, Trinh and Aldeen 1994). Indeed, we will show that the norm bounded uncertain time-delay systems can be considered as a special case of the systems considered in this paper. Moreover, it is straightforward to verify that the results of (Moheimani and Petersen 1995) can be obtained as special cases of the results presented here.

Our solution to this problem involves a finite dimensional parameter dependent algebraic Riccati equation of the game type. The number of parameters is equal to the number of uncertainty and time-delay terms. Furthermore, as in (Savkin and Petersen 1995) we show that the controller is a minimax optimal controller with respect to our Integral Quadratic Constraint uncertainty class. That is, it minimizes the maximum (over all admissible uncertainties) value of the cost function. This is expected since the problem considered in this paper is an extension of (Savkin and Petersen 1995) to the case of Integral Quadratic Constraint uncertainties which allows for time-delays. The remainder of the paper continues as follows. In Section 2, we define the class of uncertain time-delay systems under consideration. We also present our notion of absolute stability for such systems. The class of state feedback *guaranteed cost controllers* is also presented in this section. These are state feedback controllers which guarantee an upper bound on the value of a cost function corresponding to the closed loop uncertain time-delay system. Our main results are presented in Section 3 of the paper. We show that the guaranteed cost controller can be constructed by solving a parameter dependent Riccati equation. This controller guarantees absolute stability of the closed loop uncertain time-delay system. We also show that the minimax optimal controller can be determined by optimizing over the parameters entering the Riccati equation. Finally, in section 4 we present an example to illustrate how the results of this paper can be used.

2 DEFINITIONS

Consider the uncertain linear system defined by state equations

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + \sum_{p=1}^{k} C_p \xi_p(t) + \sum_{q=1}^{l} D_q \eta_q(t) \\
z_1(t) &= E_1 x(t) + F_1 u(t); \\
&\vdots \\
z_k(t) &= E_k x(t) + F_k u(t); \\
w_1(k) &= G_1 x(t) + H_1 u(t); \\
&\vdots \\
w_l(k) &= G_l x(t) + H_l u(t) \qquad (2.1)
\end{aligned}$$

where $x(t) \in \mathbf{R}^n$ is the *state*, $u(t) \in \mathbf{R}^m$ is the *control input*, $z_1(t) \in \mathbf{R}^{h_1}, \ldots, z_k(t) \in \mathbf{R}^{h_k}$ are the *uncertainty outputs*, $w_1(t) \in \mathbf{R}^{g_1}, \ldots, w_l(t) \in \mathbf{R}^{g_l}$ are the *delay uncertainty outputs*, $\xi_1(t) \in \mathbf{R}^{r_1}, \ldots, \xi_k(t) \in \mathbf{R}^{r_k}$ are the *uncertainty inputs* and $\eta_1(t) \in \mathbf{R}^{s_1}, \ldots, \eta_l(t) \in \mathbf{R}^{s_l}$ are the *delay uncertainty inputs*. Also, we assume that $C_p \neq 0$ and $D_q \neq 0$ for all $p = 1, \ldots, k$ and $q = 1, \ldots, l$. This differential equation, describes the dynamics of the system for $t \geq 0$. Moreover, we assume that $x(t) = \psi(t)$ for $t \in [-\tau, 0]$, $\psi(\cdot) \in \mathcal{L}_2[-\tau, 0]$ for a given $\tau \geq 0$ and $u(t) = 0$ for $t \leq 0$.

Associated with this system is the linear quadratic cost function

$$J = \int_0^\infty \left[x(t)' Q x(t) + u(t)' R u(t) \right] dt \qquad (2.2)$$

where $Q = Q' > 0$ and $R = R' > 0$ are given weighting matrices.

The uncertainty in the above system is described by equations of the form

$$\begin{aligned}
\xi_1(t) &= \Phi_1(t, x(\cdot)|_0^t, u(\cdot)|_0^t) \\
&\vdots \\
\xi_k(t) &= \Phi_k(t, x(\cdot)|_0^t, u(\cdot)|_0^t) \\
\eta_1(t) &= \Psi_1(t, x(\cdot)|_{-\tau_1}^t, u(\cdot)|_0^t) \\
&\vdots \\
\eta_l(t) &= \Psi_l(t, x(\cdot)|_{-\tau_l}^t, u(\cdot)|_0^t) \qquad (2.3)
\end{aligned}$$

where $\tau_q \leq \tau$ for all $q = 1, \ldots, l$. The uncertainty described by the above equation is required to satisfy a certain *Integral Quadratic Constraint*, described below.

Definition 2.1 (Integral Quadratic Constraint) Let $M_1 > 0, \ldots, M_k > 0$ and $N_1 > 0, \ldots, N_l > 0$ be given. Then an uncertainty of the form (2.3) is an admissible uncertainty for the system (2.1) if for any locally square integrable control input $u(\cdot)$ and any corresponding solution to equations (2.1), (2.3) with an interval of existence $(0,t^*)$, there exists a sequence $\{t_i\}_{i=1}^{\infty}$ such that $t_i \to t^*$, $t_i \geq 0$ and

$$\begin{aligned} \int_0^{t_i} \|\xi_p(t)\|^2 dt &\leq x(0)' M_p x(0) + \int_0^{t_i} \|z_p(t)\|^2 dt \\ \int_0^{t_i} \|\eta_q(t)\|^2 dt &\leq x(0)' N_q x(0) + \int_0^{t_i} \|w_q(t)\|^2 dt + \int_{-\tau_q}^{0} \|w_q(t)\|^2 dt \end{aligned} \tag{2.4}$$

for all i and $p = 1, 2, \ldots, k$ and $q = 1, 2, \ldots, l$. Here $\|\cdot\|$ denotes the standard Euclidean norm. Also, note that t^* and t_i may be equal to infinity. Note that the form of the Integral Quadratic Constraint on the time-delay uncertainty inputs allows the corresponding uncertain mappings $\Psi_i(\cdot)$ to include time-delays. Such an uncertain system is shown in the Figure 5.1. Notice that these Integral Quadratic Constraints pose energy constraints on each uncertainty block.

The class of uncertain systems defined by (2.1) and (2.4) is a rich uncertainty class allowing for the uncertainty inputs $\xi_p(t)$ and $\eta_q(t)$ to depend dynamically on $x(t)$ and $u(t)$. Of special interest is the class of time-delay norm bounded uncertain systems described by

$$\begin{aligned} \dot{x}(t) &= \left[A + \sum_{p=1}^{k} C_p \Delta_p(t) E_p\right] x(t) + \sum_{q=1}^{l} D_q G_q x(t-\tau_q) \\ &+ \left[B + \sum_{p=1}^{k} C_p \Delta_p(t) F_p\right] u(t) + \sum_{q=1}^{l} D_q H_q u(t-\tau_q); \quad \|\Delta_p(t)\| \leq 1 \end{aligned} \tag{2.5}$$

where $\|\Delta_p(t)\|$ are the uncertainty matrices and $\|\cdot\|$ denotes the standard induced matrix norm. To verify that such uncertainty is admissible for the uncertain system (2.1), (2.4), let $\xi_p(t) = \Delta_p(t) z_p(t)$ and $\eta_q(t) = w_q(t-\tau_q)$ where $\|\Delta_p(t)\| \leq 1$ for all $t \geq 0$. Then the uncertainty inputs $\xi_p(\cdot)$ and $\eta_q(\cdot)$ satisfy condition (2.4) with any $t_i \geq \tau$ and with any $M_p > 0$ and $N_q > 0$.

Furthermore, we can show that the uncertain system described by (2.1) and (2.4) allows for the norm bounded uncertainty to enter the time delay terms. Indeed, if we let $\xi_p(t) = \Delta_p(t) z_p(t)$ and $\eta_q(t) = \Delta_q(t) w_q(t-\tau_q)$ where $\|\Delta_p(t)\| \leq I$ and $\|\Delta_q(t)\| \leq I$, then the uncertainty inputs $\xi_p(\cdot)$ and $\eta_q(\cdot)$ satisfy condition (2.4) with any $t_i \geq \tau$ and with any $M_p > 0$ and $N_q > 0$. Hence, the following system can be considered as a special case of (2.1), (2.4)

$$\begin{aligned} \dot{x}(t) &= \left[A + \sum_{p=1}^{k} C_p \Delta_p(t) E_p\right] x(t) + \sum_{q=1}^{l} D_q \Delta_q(t) G_q x(t-\tau_q) \\ &+ \left[B + \sum_{p=1}^{k} C_p \Delta_p(t) F_p\right] u(t) + \sum_{q=1}^{l} D_q \Delta_q(t) H_q u(t-\tau_q);\ \|\Delta_p(t)\| \leq 1,\ \|\Delta_q(t)\| \leq 1 \end{aligned} \tag{2.6}$$

The uncertain system (2.5) and (2.6) include as special cases, the uncertain time-delay systems considered in the papers (Moheimani and Petersen 1995, Trinh and Aldeen 1994).

The problem considered in this paper is to optimize the worst case of the linear quadratic cost function (2.2) via a memoryless state feedback controller of the form $u(t) = Kx(t)$. When this controller is applied to the uncertain system (2.1), (2.3), it results in a closed loop uncertain system described by:

$$\begin{aligned}
\dot{x}(t) &= [A + BK]x(t) + \sum_{p=1}^{k} C_p\xi_p(t) + \sum_{q=1}^{l} D_q\eta_q(t) \\
z_1(t) &= [E_1 + F_1K]x(t); \\
&\vdots \\
z_k(t) &= [E_k + F_kK]x(t); \\
w_1(k) &= [G_1 + H_1K]x(t); \\
&\vdots \\
w_l(k) &= [G_l + H_lK]x(t) \qquad (2.7) \\
u(t) &= Kx(t) \qquad (2.8)
\end{aligned}$$

The uncertainties for this closed loop uncertain system will be described by equations of the form

$$\begin{aligned}
\xi_1(t) &= \Phi_1(t, x(\cdot)|_0^t) \\
&\vdots \\
\xi_k(t) &= \Phi_k(t, x(\cdot)|_0^t) \\
w_1(t) &= \Psi_1(t, x(\cdot)|_{-\tau_1}^t) \\
&\vdots \\
w_l(t) &= \Psi_l(t, x(\cdot)|_{-\tau_l}^t) \qquad (2.9)
\end{aligned}$$

where the integral quadratic constraint (2.4) is satisfied with the substitution $u(t) = Kx(t)$.

Definition 2.2 The controller $u(t) = Kx(t)$ is said to be a *guaranteed cost controller* for the uncertain system (2.1), (2.3) with the cost function (2.2) and $\psi(\cdot) \in \mathcal{L}_2[-\tau, 0]$ and initial condition $x(0) = x_0$ if the following conditions hold:

(i) The matrix $A + BK$ is a stability matrix.

(ii) There exists a constant $c > 0$ so that, for all admissible uncertainties, the solution to the closed loop uncertain system (2.7), (2.9) with $x(t) = \psi(t)$ for $t \in [-\tau, 0]$ and initial condition $x(0) = x_0$ is such that

$$[x(\cdot), u(\cdot), \xi_1(\cdot), \ldots, \xi_k(\cdot), \eta_1(\cdot), \ldots, \eta_l(\cdot)] \in \mathcal{L}_2[0, \infty]$$

and the corresponding value of the cost function (2.2) satisfies the bound $J \leq c$.

Note that in the above definition we are assuming that $t^* = \infty$.

Definition 2.3 The uncertain system (2.1), (2.3) with $x(t) = \psi(t)$ for $t \in [-\tau, 0]$ and initial condition $x(0) = x_0$ is called *guaranteed cost stabilizable* if it admits a guaranteed cost controller $u(t) = Kx(t)$.

Definition 2.4 The closed loop uncertain system (2.7), (2.9) is said to be *absolutely stable* if there exists constants $\alpha > 0$ and $\beta > 0$ such that the following conditions hold:

(i) For any $\psi(\cdot) \in \mathcal{L}_2[-\tau, 0]$ and initial condition $x(0) = x_0 \in \mathbf{R}^n$ and any uncertainty inputs $\xi_p(\cdot) \in \mathcal{L}_2[0, \infty)$ and $\eta_q(\cdot) \in \mathcal{L}_2[0, \infty)$, the system (2.7) has a solution $x(\cdot) \in \mathcal{L}_2[0, \infty)$.

(ii) Given any admissible uncertainties for the uncertain system (2.7), (2.9), then all corresponding solutions to equations (2.7), (2.9) satisfy $[x(\cdot), \xi_1(\cdot), \ldots, \xi_k(\cdot), \eta_1(\cdot), \ldots, \eta_l(\cdot)] \in \mathcal{L}_2[0, \infty)$ (hence, $t^* = \infty$) and

$$\|x(\cdot)\|_2^2 + \sum_{p=1}^{k} \|\xi_p(\cdot)\|_2^2 + \sum_{q=1}^{l} \|\eta(\cdot)\|_2^2 \leq \alpha\|x_0\|^2 + \beta \int_{-\tau}^{0} \|\psi(t)\|^2 dt.$$

3 ROBUST CONTROLLER SYNTHESIS

In this section, we propose a procedure to design a guaranteed cost controller for the uncertain system (2.1), (2.3). The controller minimizes the maximum value of cost function (2.2). The controller is constructed by solving a parameter dependent algebraic Riccati equation of the form

$$\begin{aligned} &(A - BE^{-1}G'L)'P + P(A - BE^{-1}G'L) \\ &\quad + P(CC' - BE^{-1}B')P + L'(I - GE^{-1}G')L = 0 \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} L &:= \begin{bmatrix} Q^{\frac{1}{2}} \\ 0 \\ \sqrt{\mu_1}E_1 \\ \vdots \\ \sqrt{\mu_k}E_k \\ \sqrt{\delta_1}G_1 \\ \vdots \\ \sqrt{\delta_l}G_l \end{bmatrix}; \quad G := \begin{bmatrix} 0 \\ R^{\frac{1}{2}} \\ \sqrt{\mu_1}F_1 \\ \vdots \\ \sqrt{\mu_k}F_k \\ \sqrt{\delta_1}H_1 \\ \vdots \\ \sqrt{\delta_l}H_l \end{bmatrix}; \quad E := G'G; \\ C &:= \begin{bmatrix} \frac{1}{\sqrt{\mu_1}}C_1 & \ldots & \frac{1}{\sqrt{\mu_k}}C_k & \frac{1}{\sqrt{\delta_1}}D_1 & \ldots & \frac{1}{\sqrt{\delta_l}}D_l \end{bmatrix}. \end{aligned} \tag{3.2}$$

We consider a set Γ defined as follows

$$\Gamma := \left\{ \begin{array}{l} [\mu_1 \ \ldots \ \mu_k \ \delta_1 \ \ldots \ \delta_k] \in \mathbf{R}^{k+l} : \mu_1 > 0, \ldots, \mu_k > 0, \delta_1 > 0, \ldots, \delta_l > 0 \\ \& \ \text{ The Riccati equation (3.1) has a solution } P > 0. \end{array} \right\}.$$

If Γ is non-empty and $P > 0$ is the minimal positive-definite solution to Riccati equation (3.1), then the corresponding guaranteed cost controller is given by

$$u(t) = -E^{-1}(B'P + G'L)x(t). \tag{3.3}$$

In the sequel we will use a certain *S-procedure* Theorem (see also (Megretsky and Treil 1993, Yakubovich 1992, Savkin and Petersen 1995)). For the sake of completeness we state this theorem here. We consider a linear system of the form:

$$\begin{aligned} \dot{\eta}(t) &= \Phi\eta(t) + \Lambda\mu(t); \quad \eta(0) = \eta_0 \\ \sigma(t) &= \Pi\eta(t) \end{aligned} \tag{3.4}$$

where Φ is a stability matrix. Corresponding to this system we consider a set $\mathcal{M}$ defined by:

$$\mathcal{M} := \left\{ \lambda(\cdot) = \begin{bmatrix} \sigma(\cdot) \\ \mu(\cdot) \end{bmatrix} : \{\sigma(\cdot), \mu(\cdot)\} \text{ are related by } (3.4), \mu(\cdot) \in \mathcal{L}_2[0,\infty) \text{ and } \eta(0) = \eta_0 \right\}. \tag{3.5}$$

Since the system (3.4) is stable, and $\mu(\cdot) \in \mathcal{L}_2[0,\infty)$, we can conclude that $\sigma(\cdot) \in \mathcal{L}_2[0,\infty)$. Therefore, $\mathcal{M} \subset \mathcal{L}_2[0,\infty)$. Also, we consider the following set of functionals mapping $\mathcal{M}$ to $\mathbf{R}$;

$$\begin{aligned} \mathcal{F}_0(\lambda(\cdot)) &:= \gamma_0 + \int_0^\infty \lambda(t)'N_0\lambda(t)dt \\ \mathcal{F}_1(\lambda(\cdot)) &:= \gamma_1 + \int_0^\infty \lambda(t)'N_1\lambda(t)dt \\ &\vdots \\ \mathcal{F}_r(\lambda(\cdot)) &:= \gamma_r + \int_0^\infty \lambda(t)'N_r\lambda(t)dt \end{aligned} \tag{3.6}$$

where N_0, N_1, ..., N_r are given matrices and γ_0, γ_1, ..., γ_r are given constants. For this set of functionals, a corresponding set $\mathcal{R} \subset \mathcal{M}$ is defined as follows:

$$\mathcal{R} := \{\lambda(\cdot) \in \mathcal{M} : \mathcal{F}_1(\lambda(\cdot)) \geq 0,\ \mathcal{F}_2(\lambda(\cdot)) \geq 0,\ ...,\ \mathcal{F}_r(\lambda(\cdot)) \geq 0\}.$$

The following lemma is referred to as the *S-procedure* Theorem (see (Megretsky and Treil 1993),(Yakubovich 1992))

Lemma 3.1 *Consider a system of the form (3.4), a set $\mathcal{M}$ of the form (3.5) and a set of functionals of the form (3.6). Suppose that these functionals have the following properties:*

(i) $\mathcal{F}_0(\lambda(\cdot)) \leq 0$ for all $\lambda(\cdot) \in \mathcal{R}$;

(ii) There exists a $\lambda(\cdot) \in \mathcal{M}$ such that $\mathcal{F}_1(\lambda(\cdot)) > 0$, $\mathcal{F}_2(\lambda(\cdot)) > 0$, ..., $\mathcal{F}_r(\lambda(\cdot)) > 0$.

Then there will exist constants $\beta_1 \geq 0$, $\beta_2 \geq 0$, ... , $\beta_r \geq 0$ such that

$$\mathcal{F}_0(\lambda(\cdot)) + \sum_{i=1}^r \beta_i\mathcal{F}_i(\lambda(\cdot)) \leq 0$$

for all $\lambda(\cdot) \in \mathcal{M}$.

The following theorem gives a characterization of guaranteed cost controllers in terms of the parameter dependent Riccati equation (3.1). It also determines the corresponding upper bound on the cost function (2.2) with this controller.

Theorem 3.1 *Consider the uncertain delay system (2.1), (2.4) with cost function (2.2). Then for any* $\{\mu_1, \dots, \mu_k, \delta_1, \dots, \delta_l\} \in \Gamma$*, the corresponding controller (3.3) is a guaranteed cost controller for this uncertain delay system with any* $\psi(\cdot) \in \mathcal{L}_2[-\tau, 0]$ *and initial condition* $x_0 \in \mathbf{R}^n$*. Furthermore, the corresponding value of the cost function (2.2) satisfies the bound*

$$J \leq x_0'\{P + \sum_{p=1}^{k} \mu_p M_p + \sum_{q=1}^{l} \delta_q N_q\} x_0 + \sum_{q=1}^{l} \delta_q \int_{-\tau_q}^{0} \psi(t)' G_q' G_q \psi(t) dt \tag{3.7}$$

for all admissible uncertainties. Moreover, the closed loop uncertain delay system (2.1), (2.4), (3.3) is absolutely stable.

Conversely, with Ω *the set of admissible uncertainties, if the uncertain system (2.1), (2.4) with the cost function (2.2) and* $\psi(\cdot) \in \mathcal{L}_2[-\tau, 0]$ *and initial condition* $x(0) = x_0$ *has a guaranteed cost controller such that*

$$\sup_{[\xi(\cdot), \eta(\cdot)] \in \Omega} J < c_1, \tag{3.8}$$

then there exist $[\mu, \delta] \in \Gamma$ *such that*

$$x_0'\{P + \sum_{p=1}^{k} \mu_p M_p + \sum_{q=1}^{l} \delta_q N_q\} x_0 + \sum_{q=1}^{l} \delta_q \int_{-\tau_q}^{0} \psi(t)' G_q' G_q \psi(t) dt < c_1. \tag{3.9}$$

Proof: Let $[\mu_1 \ \dots \ \mu_k \ \delta_1 \ \dots \ \delta_l] \in \Gamma$ be given and let $P > 0$ be the corresponding positive definite solution of the Riccati equation (3.1). Also, assume that the initial condition of the uncertain system (2.1), (2.3) is defined by $x(t) = \psi(t) \in \mathcal{L}_2[-\tau, 0]$ and $x(0) = x_0 \in \mathbf{R}^n$. If controller (3.3) is applied to the uncertain system (2.1), (2.3) the closed loop system will be defined by (2.7) with $K = -E^{-1}(B'P + G'L)$. Now, for a given initial condition, define a functional $\mathcal{F}$ as follows

$$\mathcal{F}(x(\cdot), u(\cdot), z(\cdot), w(\cdot), \xi(\cdot), \eta(\cdot)) = J + \sum_{p=1}^{k} \mu_p (\|z_p(\cdot)\|_2^2 - \|\xi_p(\cdot)\|_2^2) + \sum_{q=1}^{l} \delta_q (\|w_q(\cdot)\|_2^2 - \|\eta_q(\cdot)\|_2^2). \tag{3.10}$$

Also, let us consider a differential game where the underlying system is described by the state equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + C\nu(t); \quad x(0) = x_0 \\ \zeta(t) &= Lx(t) + Gu(t) \end{aligned} \tag{3.11}$$

and the cost function is given by

$$\mathcal{L}(u(\cdot), \nu(\cdot)) = \int_0^\infty (\|\zeta(t)\|^2 - \|\nu(t)\|^2) dt \tag{3.12}$$

where C, L and G are defined as in (3.2) and $\nu(\cdot) = [\sqrt{\mu_1}\xi_1(\cdot)' \ \ldots \ \sqrt{\mu_k}\xi_k(\cdot)' \ \ldots \ \sqrt{\delta_1}\eta_1(\cdot)' \ \ldots \ \sqrt{\delta_l}\eta_l(\cdot)']'$. In this game, $u(t)$ is the *minimizing player input*, $\nu(t)$ is the *maximizing player input* and $\zeta(t)$ is the output of the system defined by (3.11). It is easy to verify that the system (3.11) can be written as (2.1) and the functional $\mathcal{L}$ is equivalent to the functional $\mathcal{F}$ defined in (3.10).

Theorem 4.8 of (Basar and Bernhard 1991) implies that

$$\sup_{\nu(\cdot)\in\mathcal{L}_2[0,\infty)} \mathcal{L}\left(u(\cdot),\nu(\cdot)\right) = x_0'Px_0$$

where P is the minimal positive definite solution of the Riccati equation (3.1). Furthermore, the controller which achieves this supremum is defined as in (3.3). Therefore,

$$\sup_{[\xi(\cdot),\eta(\cdot)]\in\mathcal{L}_2[0,\infty)} \mathcal{F}\left(x(\cdot),u(\cdot),z(\cdot),w(\cdot),\xi(\cdot),\eta(\cdot)\right) = x_0'Px_0 \tag{3.13}$$

Moreover, it follows from Theorem 4.8 and Section 4.5.1 of (Basar and Bernhard 1991) that the matrix $(A-BE^{-1}[B'P+G'L])$ is stable. That is, the system (2.7) with K defined as in (3.3) is asymptotically stable.

Now, we prove that for the closed loop system described by equations (2.7) and (3.3), $t^* = \infty$. Indeed, for an admissible uncertainty let sequence $\{t_j\}_{j=1}^{\infty}$ be as defined in Definition 2.1 and consider a corresponding sequence $\lambda^j(\cdot) = [x^j(\cdot),u^j(\cdot),z^j(\cdot),w^j(\cdot),\xi^j(\cdot),\eta^j(\cdot)]$ of vector functions defined by the initial condition $x^j(0) = x_0$ and inputs $\xi^j(\cdot)$ and $\eta^j(\cdot)$ defined as:

$$\xi^j(t) = \begin{cases} \xi^0(t) & \text{for } 0 < t < t_j; \\ 0 & \text{for } t \geq t_j \end{cases} ; \quad \eta^j(t) = \begin{cases} \eta^0(t) & \text{for } 0 < t < t_j; \\ 0 & \text{for } t \geq t_j \end{cases} .$$

It is clear that $\lambda^j(t) = \lambda^0(t)$ for $0 < t < t_j$. Also, since the matrix $(A - BE^{-1}[B'P + G'L])$ is stable, we conclude that $\lambda^j(\cdot) \in \mathbf{L}_2[0,\infty)$. Therefore, (3.13) implies that

$$\begin{aligned} x_0'Px_0 \ \geq \ & J(x^j(\cdot),u^j(\cdot)) + \sum_{p=1}^{k} \mu_p(\|z_p^j(\cdot)\|_2^2 - \|\xi_p^j(\cdot)\|_2^2) \\ & + \sum_{q=1}^{l} \delta_q(\|w_q^j(\cdot)\|_2^2 - \|\eta_q^j(\cdot)\|_2^2). \end{aligned} \tag{3.14}$$

This, combined with inequality (2.4) implies that

$$J(x^j(\cdot),u^j(\cdot)) \leq x_0'\{P + \sum_{p=1}^{k} \mu_p M_p + \sum_{q=1}^{l} \delta_q N_q\}x_0 + \sum_{q=1}^{l} \delta_q \int_{-\tau_q}^{0} x(t)'G_q'G_q x(t)dt \tag{3.15}$$

for all j. Now as in Definition 2.1 we have $t_j \to t^*$. However, since $Q > 0$, (3.15) implies that $t^* \to \infty$. Therefore, we conclude that $\lambda^0(\cdot) \in \mathcal{L}_2[0,\infty)$. Also, (3.15) implies that (3.7) is satisfied.

To complete the proof of the first part of the theorem we have to establish absolute stability of the closed loop uncertain system (2.7), (2.9) as defined by Definition 2.4. We have already proved that the matrix $(A - BE^{-1}[B'P + G'L])$ is stable. This establishes the condition (i) of definition (2.4). To prove the condition (ii), note that we have already proved that $[x(\cdot), \xi_1(\cdot), \dots, \xi_k(\cdot), \eta_1(\cdot), \dots, \eta_l(\cdot)] \in \mathcal{L}_2[0, \infty]$. Also, inequality (3.7) implies that there exist constants $c_1, c_2 > 0$ such that $\|x(\cdot)\|_2^2 \leq c_1\|x_0\|^2 + c_2 \int_{-\tau}^0 \|\psi(\cdot)\|^2 dt$ for all the solutions to the closed loop uncertain system. Furthermore, the constraint (2.4) imply that there exist constants $c_3, c_4, c_5, c_6, c_7 > 0$ such that $\|\xi_p(\cdot)\|_2^2 \leq c_3\|x(\cdot)\|_2^2 + c_4\|x_0\|^2$ and $\|\eta_q(\cdot)\|_2^2 \leq c_5\|x(\cdot)\|_2^2 + c_6\|x_0\|^2 + c_7 \int_{-\tau}^0 \|\psi(\cdot)\|^2 dt$ for all the solutions of the closed loop uncertain system and all $p = 1, \dots, k$ and $q = 1, \dots, l$. Hence,

$$\|x(\cdot)\|_2^2 + \sum_{p=1}^k \|\xi_p(\cdot)\|_2^2 + \sum_{q=1}^l \|\eta(\cdot)\|_2^2 \leq [(1 + kc_3 + lc_5)c_1 + kc_4 + lc_6]\|x_0\|^2 + (c_2 + lc_7)\int_{-\tau}^0 \|\psi(\cdot)\|^2 dt.$$

This completes the proof of the first part of the theorem.

To prove the second part of the theorem, let $\psi(\cdot) \in \mathcal{L}_2[-\tau, 0]$ be given such that $x(0) = x_0 \neq 0$. Let us define functionals $\mathcal{F}_1, \dots, \mathcal{F}_k$ and $\mathcal{G}_1, \dots, \mathcal{G}_l$ as follows

$$\begin{aligned} \mathcal{F}_p(\lambda(\cdot)) &= \|z_p(\cdot)\|_2^2 - \|\xi(\cdot)\|_2^2 + x_0' M_p x_0 \geq 0; & (3.16) \\ \mathcal{G}_q(\lambda(\cdot)) &= \|w_q(\cdot)\|_2^2 - \|\eta(\cdot)\|_2^2 + \int_{-\tau_q}^0 \|w_q(t)\|^2 dt + x_0' N_q x_0 \geq 0 & (3.17) \end{aligned}$$

where

$$\lambda(\cdot) = [x(\cdot), u(\cdot), z(\cdot), w(\cdot), \xi(\cdot), \eta(\cdot)].$$

Note that (3.8) implies that there exist a constant $\epsilon > 0$ such that $(1 + \epsilon)J(x(\cdot), u(\cdot)) \leq c_1 - \epsilon$. Now, if we define a functional $\mathcal{F}_0$ as

$$\mathcal{F}_0(\lambda(\cdot)) = (1 + \epsilon)J(x(\cdot), u(\cdot)) - c_1 + \epsilon \leq 0, \tag{3.18}$$

we can apply the *S-procedure* Theorem to the functionals (3.16), (3.17) and (3.18). The S-procedure implies that there exist constants $\mu_1 \geq 0, \dots, \mu_k \geq 0$ and $\delta_1 \geq 0, \dots, \delta_l \geq 0$ such that

$$\mathcal{F}_0(\lambda(\cdot)) + \sum_{p=1}^k \mu_p \mathcal{F}_p(\lambda(\cdot)) + \sum_{q=1}^l \delta_q \mathcal{G}_q(\lambda(\cdot)) \leq 0. \tag{3.19}$$

Let us define the functional $\mathcal{F}_\epsilon$ as

$$\mathcal{F}_\epsilon(\lambda(\cdot)) = \epsilon J(x(\cdot), u(\cdot)) + \mathcal{F}(\lambda(\cdot)) \tag{3.20}$$

with $\mathcal{F}$ defined as in (3.10). With this definition, (3.19) implies that

$$\mathcal{F}_\epsilon(\lambda(\cdot)) \leq c_1 - \epsilon - x_0'\{P + \sum_{p=1}^k \mu_p M_p + \sum_{q=1}^l \delta_q N_q\}x_0 - \sum_{q=1}^l \delta_q \int_{-\tau_q}^0 \psi(t)' G_q' G_q \psi(t) dt. \tag{3.21}$$

Let $\lambda(\cdot) = [x(\cdot), u(\cdot), z(\cdot), w(\cdot), \xi(\cdot), \eta(\cdot)] \in \mathcal{L}_2[0, \infty)$ be the solution of (2.7) corresponding to $x(0) = 0$. Also, let $\lambda_0(\cdot) = [x_0(\cdot), u_0(\cdot), z_0(\cdot), w_0(\cdot), 0, 0]$ be the solution of (2.7) corresponding to the initial condition $x(0) = x_0$ and $\xi(\cdot) \equiv 0$ and $\eta(\cdot) \equiv 0$. Since the system (2.7) is linear, $a\lambda + \lambda_0$ is also a solution of (2.7) with $x(0) = x_0$ for all $a \in \mathbf{R}$. Furthermore, since $\mathcal{F}_\epsilon$ is a quadratic functional, we can write

$$\mathcal{F}_\epsilon(a\lambda + \lambda_0) = a^2\mathcal{F}_\epsilon(\lambda) + af(\lambda, \lambda_0) + \mathcal{F}_\epsilon(\lambda_0)$$

where $f(\cdot, \cdot)$ is a corresponding bilinear form. Now, if we assume that $\mathcal{F}_\epsilon(\lambda) > 0$, it follows that $\lim_{a\to\infty} \mathcal{F}_\epsilon(a\lambda + \lambda_0) \to \infty$ which contradicts (3.21). Therefore, $\mathcal{F}_\epsilon(\lambda(\cdot)) \leq 0$ for all solutions of (2.7) corresponding to $x(0) = 0$.

Next, we prove that $\mu_p, \delta q > 0$ for all $p = 1, \dots, k$ and $q = 1, \dots, l$. Note that $\mathcal{F}_\epsilon(\lambda(\cdot))$ can be written as

$$\mathcal{F}_\epsilon(\lambda(\cdot)) = \int_0^\infty \left\{ \begin{array}{l} (1+\epsilon)[x(t)'Qx(t) + u(t)'Ru(t)] \\ +\sum_{p=1}^k \mu_p(\|z_p(\cdot)\|^2 - \|\xi_p(\cdot)\|^2) + \sum_{q=1}^l \delta_q(\|w_q(\cdot)\|^2 - \|\eta_q(\cdot)\|^2) \end{array} \right\} dt \qquad (3.22)$$

for all $[x(\cdot), u(\cdot), z(\cdot), w(\cdot), \xi(\cdot), \eta(\cdot)] \in \mathcal{L}_2[0, \infty)$ connected by (2.7) with $x(0) = 0$. Let us assume that $\mu_i = 0$, $\xi_i(\cdot) \neq 0$ and $\xi_p(\cdot) \equiv 0$ for $p \neq i$ for some i. Also, let $\delta_j(\cdot) = 0$, $\eta_j(\cdot) \neq 0$ and $\eta_q(\cdot) \equiv 0$ for $q \neq j$ for some j. For such an input, (3.22) implies that $\mathcal{F}_\epsilon(\lambda(\cdot)) = 0$ since we have already proved that $\mathcal{F}_\epsilon(\lambda(\cdot)) \leq 0$ for any solution of the system (2.7) with zero initial condition. This along with the assumption that $Q > 0$ and $R > 0$, imply that $x(\cdot) = 0$ and $u(\cdot) = 0$. However, since $C_i \neq 0$ and $D_j \neq 0$ we can choose $\xi_i(\cdot)$ and $\eta_j(\cdot)$ such that $C_i\xi_i(\cdot) \neq 0$ and $D_j\eta_j(\cdot) \neq 0$. This is in contradiction with equation (2.7). Hence, the logical conclusion is that $\mu_p, \delta_q > 0$ for all $p = 1, \dots, k$ and $q = 1, \dots, l$.

Now, consider the functional $\mathcal{F}_\epsilon(\lambda(\cdot))$ defined above. We have already shown that $\mathcal{F}_\epsilon(\lambda(\cdot)) \leq 0$ for any $\lambda(\cdot) = [x(\cdot), u(\cdot), z(\cdot), w(\cdot), \xi(\cdot), \eta(\cdot)] \in \mathcal{L}_2[0, \infty)$ connected by (2.7) with initial condition $x(0) = 0$. Therefore, (3.20) implies that $\mathcal{F}(\lambda(\cdot)) \leq -\epsilon J$. Now consider the system (3.11) with $x(0) = 0$. Since $\mu_p, \delta_q > 0$, this system is well-defined. If we let $\nu(\cdot)$ be the disturbance input and $\zeta(\cdot)$ be the controlled output, it follows that $\mathcal{F}$ as defined by (3.10) can be rewritten as

$$\mathcal{F}(u(\cdot), \nu(\cdot)) = \|\zeta(\cdot)\|_2^2 - \|\nu(\cdot)\|_2^2.$$

Since we have established that $\mathcal{F} < 0$, we may conclude that

$$J_\infty = \sup_{x(0)=0, \nu(\cdot)\in\mathcal{L}_2[0,\infty)} \frac{\|\zeta(\cdot)\|_2^2}{\|\nu(\cdot)\|_2^2} < 1. \qquad (3.23)$$

This means that the controller (3.3) solves a standard $\mathcal{H}_\infty$ control problem defined by the system (3.11) with initial condition $x(0) = 0$ and the cost function J_∞ defined by (3.23). Hence, using a standard result from $\mathcal{H}_\infty$ control theory, it follows that the Riccati equation (3.1) has a positive definite solution $P > 0$, e.g., see Theorem 4.11 of (Basar and Bernhard 1991).

To conclude, consider the differential game defined by system (3.11) and cost function (3.12). Also, assume that $x(0) = x_0 \in \mathbf{R}^n$. We have already proved that the Riccati equation (3.1) has a positive definite solution $P > 0$. Therefore, Theorem 4.8 and Section 4.5.1 of (Basar and Bernhard 1991) imply that $\sup_{\nu(\cdot)\in\mathcal{L}_2[0,\infty)} \mathcal{L} \geq x_0' P x_0$. Hence,

$$\sup_{[\xi(\cdot),\eta(\cdot)]\in\mathcal{L}_2[0,\infty)} \mathcal{F}(u(\cdot),\nu(\cdot)) \geq x_0' P x_0.$$

This last inequality along with (3.21) imply that

$$\begin{aligned} c_1 - \epsilon - x_0'\{\sum_{p=1}^{k}\mu_p M_p + \sum_{q=1}^{l}\delta_q N_q\}x_0 - \sum_{q=1}^{l}\delta_q\int_{-\tau_q}^{0}\psi(t)'G_q'G_q\psi(t)dt &\geq \sup_{[\xi(\cdot),\eta(\cdot)]\in\mathcal{L}_2[0,\infty)} \mathcal{F}_\epsilon(\lambda(\cdot)) \\ &\geq \sup_{[\xi(\cdot),\eta(\cdot)]\in\mathcal{L}_2[0,\infty)} \mathcal{F}(\lambda(\cdot) \\ &\geq x_0' P x_0 \end{aligned}$$

Therefore,

$$x_0'\{P + \sum_{p=1}^{k}\mu_p M_p + \sum_{q=1}^{l}\delta_q N_q\}x_0 + \sum_{q=1}^{l}\delta_q\int_{-\tau_q}^{0}\psi(t)'G_q'G_q\psi(t)dt \leq c_1 - \epsilon$$

which implies that the inequality (3.9) is true. This completes the proof of the second part of the theorem. $\diamond\diamond\diamond$

The next theorem is the main result of this paper and shows that Theorem 3.1 can be used to construct a controller which approaches the minimax optimum. The proof of the theorem follows directly from the statements of Theorem 3.1.

Theorem 3.2 *Consider the uncertain delay system (2.1), (2.3) with cost function (2.2), and assume that* $C_1 \neq 0,\ldots,C_k \neq 0$ *and* $D_1 \neq 0,\ldots,D_l \neq 0$. *Then:*

(i) Given a $\psi(\cdot) \in \mathcal{L}_2[0,\infty)$ *with a non-zero initial condition* $x(0) \in \mathbf{R}^n$, *the uncertain delay system (2.1), (2.3) will be guaranteed cost stabilizable with initial condition* $x(0) = x_0$ *if and only if the set* Γ *is not empty.*

(ii) Suppose the set Γ *is not empty and let* Ω *be the set of all admissible uncertainties for the uncertain system (2.1), (2.3) as defined in definition 2.1. Also, for any initial condition* $x_0 \neq 0$, *let* Θ *denote the set of all guaranteed cost controllers of the form (3.3) for the uncertain system with this initial condition. Then*

$$\inf_{u(\cdot)\in\Theta}\sup_{[\xi(\cdot),\eta(\cdot)]\in\Omega} J = \inf_{[\delta,\mu]\in\Gamma}\left\{x_0'\{P + \sum_{p=1}^{k}\mu_p M_p + \sum_{q=1}^{l}\delta_q N_q\}x_0 + \sum_{q=1}^{l}\delta_q\int_{-\tau_q}^{0}\psi(t)'G_q'G_q\psi(t)dt\right\}. \tag{3.24}$$

Remark: Note that our state feedback minimax optimal controller depends on the actual values of time-delay terms $\tau_1,\ldots,\tau_l$ since the optimal parameters $\mu_1,\ldots,\mu_k$ and $\delta_1,\ldots,\delta_l$ depend on the time delays. Therefore, the minimax optimal controller proposed in this paper is indeed a delay dependent controller. However, if the controller is designed to guarantee only the robust stability of the system, and not the minimax optimality, then such a controller would not be delay dependent.

Remark: For the case of a single uncertain parameter and time-delay in the state, the Riccati equation (3.1) reduces to the Riccati equation of (Moheimani and Petersen 1995). Also, the controller (3.3) is identical to the controller proposed in (Moheimani and Petersen 1995). Furthermore, if we let $M_p \to 0$ and $N_q \to 0$ for all p and q, then the same cost bound is obtained as well.

Remark: Note that the controller designed using the above procedure is nonconservative if the uncertainties are precisely modeled by the integral quadratic constraints (2.3). However, if the time delays are known and modelled using the IQC's, then the controller will no longer be minimax optimal. But, it will be a guaranteed cost controller which minimizes a bound on the LQ cost function. This will cause a certain amount of conservativeness since it may not be clear how tight that bound is.

Remark: The minimax optimal controller of this paper is a state feedback controller. The problem of designing a minimax output feedback controller within this framework is more complicated, and hence, delegated to future research. However, it is possible to modify the IQC descriptions of uncertainty in (Savkin and Petersen 1994*c*) and (Savkin and Petersen 1994*b*) to allow for time delay terms and derive the corresponding output feedback controllers.

As stated in Theorem 3.2, existence of a minimax optimal controller is guaranteed if the set Γ is nonempty. The following lemma gives necessary and sufficient conditions for existence of a nonempty set Γ in terms of feasibility of a linear matrix inequality.

Lemma 3.2 *The following statements are equivalent:*
(i) The set Γ is nonempty.
(ii) The following Linear Matrix Inequality is feasible:

$$\begin{bmatrix} \begin{pmatrix} AZ+ZA'+BY+Y'B' \\ +\sum_{p=1}^{k}\frac{1}{\mu_p}C_pC_p' \\ +\sum_{q=1}^{l}\frac{1}{\delta_q}D_qD_q' \end{pmatrix} & (E_1Z+F_1Y)' & \dots & (E_kZ+F_kY)' & (G_1Z+H_1Y)' & \dots & (G_lZ+H_lY)' & Z & Y' \\ (E_1Z+F_1Y) & -\frac{1}{\mu_1}I & & & & & & & \\ \vdots & & \ddots & & & & & & \\ (E_kZ+F_kY) & & & -\frac{1}{\mu_k}I & & & & & \\ (G_1Z+H_1Y) & & & & -\frac{1}{\delta_1}I & & & & \\ \vdots & & & & & \ddots & & & \\ (G_lZ+H_lY) & & & & & & -\frac{1}{\delta_l}I & & \\ Z & & & & & & & -Q^{-1} & \\ Y & & & & & & & & -R^{-1} \end{bmatrix} < 0. \tag{3.25}$$

Proof: Let $K = -E^{-1}(B'P + G'L)$. Then it is straightforward, but tedious, to show that the Riccati equation (3.1) is equivalent to

$$(A+BK)'P + P(A+BK) + PCC'P + (L+GK)'(L+GK) = 0.$$

It follows from the Strict Bounded Real Lemma (Theorem 2.1 of (Petersen *et al.* 1991)) that the above Riccati equation has a solution if and only if there exists a $\tilde{P} > 0$ such that

$$(A+BK)'\tilde{P} + \tilde{P}(A+BK) + \tilde{P}CC'\tilde{P} + (L+GK)'(L+GK) < 0. \tag{3.26}$$

Using (3.2) and the change of variable $K = Y\tilde{P}^{-1} = YZ$, it is straightforward to show that (3.25) follows from (3.26) using the standard Schur complement.

4 ILLUSTRATIVE EXAMPLE

In this section we present an example to illustrate our approach to minimax stabilization of uncertain time-delay systems. We consider an uncertain time-delay system described by the following delay differential equations:

$$\begin{aligned}
\dot{x}_1(t) &= [3.4 - 0.5\Delta(t)]x_1(t) + [-1.1 - 0.5\Delta(t)]x_2(t) - u(t); \\
\dot{x}_2(t) &= [0.5 + 1.1\Delta(t)]x_1(t) + 0.5x_1(t-1) + [-2 + \Delta(t)]x_2(t) + 0.25x_2(t-1) + 2u(t)
\end{aligned}$$

where $\Delta(t)$ is a scalar uncertain parameter satisfying the bound $|\Delta(t)| \leq 1$. We also assume that $\psi(t) = [1.2 \quad 0]'$ for $t \in [-1, 0]$. Corresponding to this system we consider the linear quadratic cost function

$$J = \int_0^\infty [x_1(t)^2 + x_2(t)^2 + u(t)^2]dt.$$

This system and the corresponding LQ cost function are of the form (2.1), (2.3), (2.2) with

$$\begin{aligned}
A &= \begin{bmatrix} 3.4 & -1.1 \\ 0.5 & -2 \end{bmatrix}; \; B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}; \; C_1 = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}; \\
D_1 &= \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}; \; M_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}; \; N_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}; \\
Q &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \; E_1 = \begin{bmatrix} 1.1 \\ 1 \end{bmatrix}'; \; G_1 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}'; \; R = 1.
\end{aligned}$$

The matrices M_1 and N_1 can be regarded as bounding the initial conditions on the uncertainty dynamics. The reason for choosing them as above is to reflect the assumption that the initial conditions will be small in this example.

We now apply the results of the previous section to this uncertain system and cost function. This involves solving the Riccati equation (3.1) for a series of the parameters μ and δ. A plot of the cost J versus μ and δ is shown in Figure 5.2. From this figure, we can see that the optimal value of the parameters μ and δ is $\mu = 0.95$ and $\delta = 0.16$. With this values of μ and δ we find the positive definite solution to the Riccati equation (3.1);

$$P = \begin{bmatrix} 8.1614 & -0.3419 \\ -0.3419 & 0.4694 \end{bmatrix} > 0.$$

The following minimax optimal controller is found using equation (3.3)

$$u(t) = [8.8451 \quad -1.2807]x(t).$$

To verify the robust stability of this control system, Figure 5.3 shows the evolution of the closed loop system states $x_1(t)$ and $x_2(t)$ for values of uncertainty ranging between -1 and 1.

5 Conclusion

In this paper we looked at the problem of designing robust state feedback controllers for a class of uncertain time-delay systems where the uncertainty is described by an Integral Quadratic Contraint. It was explained how a minimax optimal controller could be designed for such a system by solving a parameter dependent Riccati equation. Moreover, it was explained how the existance of a minimax optimal controller can be checked by solving a particular Linear Matrix Inequality.

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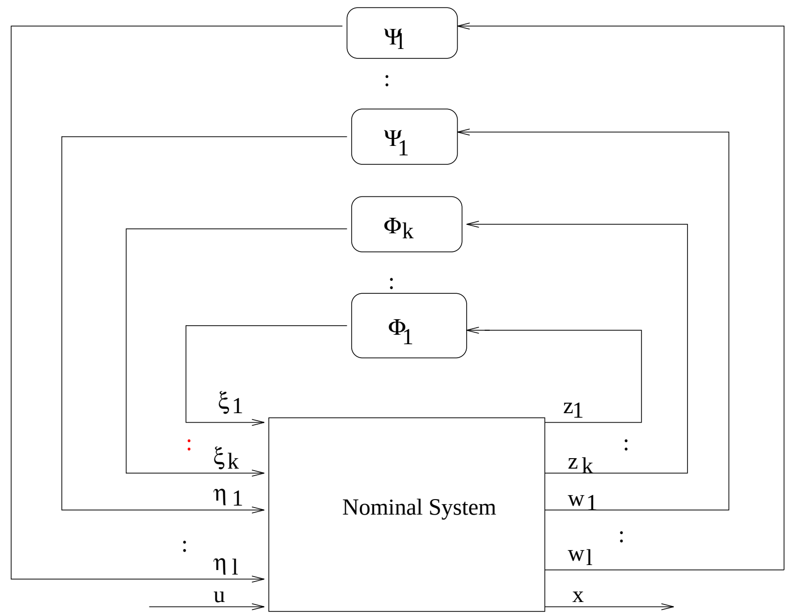

Figure 5.1: Block diagram of an uncertain system with Integral Quadratic Constraint uncertainties.

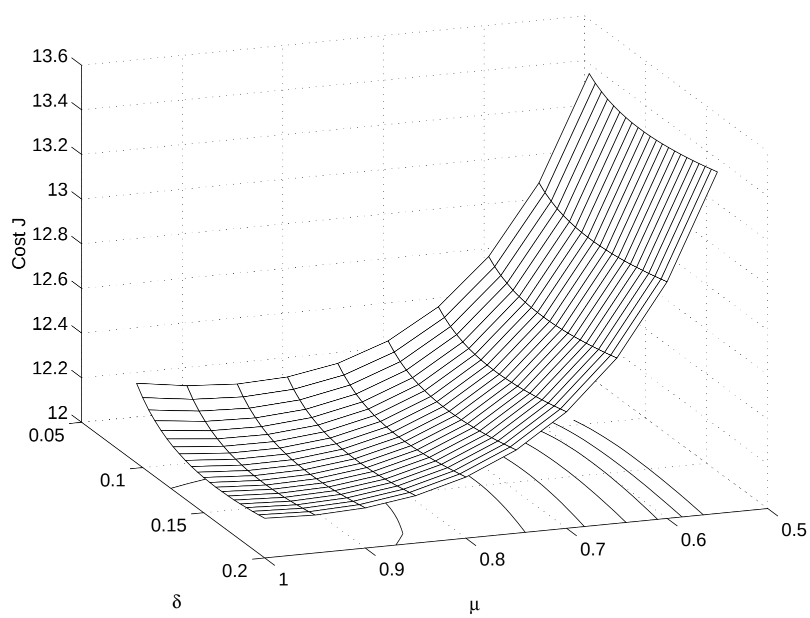

Figure 5.2: $x_0'\{P + \mu M_1 + \delta N_1\}x_0 + \delta \int_{-1}^{0} \psi(t)' G_1' G_1 \psi(t) dt$ versus the parameters μ and δ.

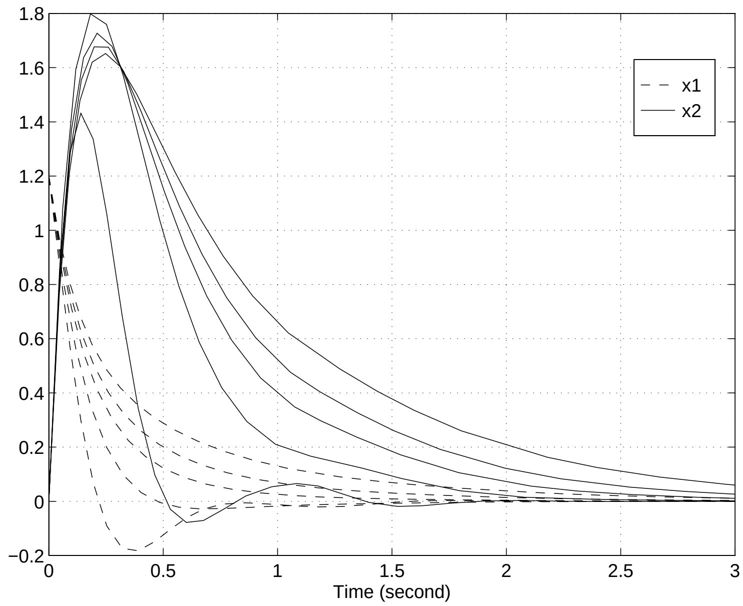

Figure 5.3: States of the system versus time for uncertainty $-1 \leq \Delta \leq 1$.