Minimizing the truncation error in assumed modes models of structures

REVISED VERSION

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Abstract

The assumed modes approach is a widely used technique in modeling of distributed systems. Such models often consist of a large number of modes. For controller design purposes these models are simplified by truncating the modes that lie out of the bandwidth of interest. Truncation can alter zeros of the system. This paper presents a method of minimizing the truncation error by adding a second order term to the truncated model. This extra term is determined such that the in-bandwidth error is minimized in an optimal $\mathcal{H}_2$ sense. The technique is extended to multivariable systems.

1 Introduction

The assumed modes approach has been used extensively throughout the literature to model dynamics of distributed systems. Such systems include, but are not limited to, flexible beams and plates [1], slewing beams [2], piezoelectric laminate beams [3] and acoustic ducts [4]. Dynamics of each one of these systems is described by a particular partial differential equation. In the assumed modes approach, the solutions of these partial differential equations are practically based upon a finite set of terms in the expansion; however one is always faced with the tradeoff between model order (number of terms) and model fidelity (convergence of the solution).

In control design problems, one is often only interested in designing a controller for a particular frequency range. In these situations, it is common practice to remove the modes which correspond to frequencies that lie out of the bandwidth of interest and only keep the modes which directly contribute to the low frequency dynamics of the system. This model is then used to design a controller. The performance predicted by such models and control system designs typically exceed the practical performance which can be achieved in the laboratory. This is mainly due to the fact that although the poles of the truncated system are at the correct frequencies, the zeros can be far away from where they should

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be. Therefore, it is natural to expect significant differences between predicted performance based upon truncated models and achieved performance based upon experimental systems since the closed loop performance of the system is largely dictated by the open loop zeros.

In a recent paper, the second author demonstrated the effect of out of bandwidth zeros on the low frequency zeros of the truncated model [5]. He also suggested that the effect of higher frequency modes on the low frequency dynamics of the system can be captured by adding a zero frequency term to the truncated model to account for the compliance of the ignored modes.

In [6], the first author showed that the effect of truncated modes on the low-frequency dynamics of the structure can be captured in an optimal way. In particular, [6] shows how a feed-through term that minimizes the $\mathcal{H}_2$ norm of the error system can be found. Moreover, the same procedure is extended to multivariable structural models.

In this paper, we extend the works of [5] and [6] to allow for the effect of truncated modes to be captured by a second order term. Moreover, we will attempt to determine the resonant frequency and the gain of this term in a way that the $\mathcal{H}_2$ norm of the error system is minimized.

2 Problem statement

Let us consider the transfer function of a flexible structure:

$$G(s) = \sum_{i=1}^{\infty} \frac{F_i}{s^2 + \omega_i^2}. \tag{1}$$

In a typical control design scenario, the designer is often interested only in a particular bandwidth. Therefore, an approximate model of the system is needed that best represents the dynamics of the system in a particular frequency range. A natural choice in this case is to simply ignore the modes which correspond to the frequencies that lie out of the bandwidth of interest. For instance, if ω_N is equivalent or larger than the highest frequency of interest, one may choose to approximate $G(s)$ by $G_N(s) = \sum_{i=1}^{N} \frac{F_i}{s^2+\omega_i^2}$. A drawback of this approach is that the ignored higher order modes may contribute to the low frequency dynamics in the form of distorting zero locations. In reference [5], the second author suggested a way of dealing with this problem. The idea that was put forward in [5] is to allow for a constant feed-through term in $G_N(s)$ to account for the compliance of omitted higher order modes of (1). That is, to approximate $G(s)$ by $\hat{G}(s) = G_N(s) + K$ where $K = \sum_{i=N+1}^{\infty} \frac{F_i}{\omega_N^2}$. The logic behind this choice of K is that at lower frequencies, one can ignore the effect of dynamical response of higher order modes since they are small in comparison with the force responses. Although an approximation, reference [5] shows that K is a good representation of the effect of higher order modes on $G_N(s)$. Indeed, it can be shown that this choice of K brings the error to zero at $\omega = 0$. In reference [6], the first author showed that K can be chosen in a way that the $\mathcal{H}_2$ norm of the error system, i.e., $G(s) - G_N(s)$ is minimized in the bandwidth of interest. It was also shown that this choice of K could result in a higher error at $\omega = 0$. However, the error at higher frequencies within the bandwidth of interest is lower with this choice of K.

In this paper we choose to replace the constant feed-through term with a second order resonant term whose resonance frequency lies out of the bandwidth of interest. In other words, we try to capture the effect of out of bandwidth modes on the in-bandwidth dynamics of the truncated model using a second order system such as: $M(s) = \frac{K}{s^2+\alpha^2}$ where $\alpha > \omega_c$ and ω_c is the highest frequency of interest. Hence, we approximate $G(s)$ by

$$\hat{G}(s) = G_N(s) + M(s). \tag{2}$$

Our objective is to choose K and α such that the following cost function is minimized,

$$e(K,\alpha) = \|(G(s) - \hat{G}(s))W(s)\|_2^2. \tag{3}$$

Here $\|f(s)\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(j\omega)\|^2 d\omega$ where $f(s)$ is a rational function. Moreover, G and $\hat{G}$ are defined as in (1) and (2) and $W(s)$ is an ideal low-pass weighting function with its cut-off frequency ω_c chosen to lie within the interval $\omega_c \in (\omega_N, \omega_{N+1})$. That is, $|W(j\omega)| = 1$ for $-\omega_c \leq \omega \leq \omega_c$ and zero elsewhere. To this end, it should be clear that K and α chosen to minimize (3), will minimize the effect of out of bandwidth dynamics of $G(s)$ on $\hat{G}(s)$ in an $\mathcal{H}_2$ optimal sense. It is easy to see that (3) is equivalent to

$$e(K,\alpha) = \left\| \left(\sum_{i=N+1}^{\infty} \frac{F_i}{s^2+\omega_i^2} - \frac{K}{s^2+\alpha^2} \right) W(s) \right\|_2^2. \tag{4}$$

The fact that W is chosen to be an ideal low-pass filter with its cut-off frequency lower than the first out-of-bandwidth pole of G, guarantees that (4) will remain finite. It is straightforward to show that (4) is equivalent to

$$e(K,\alpha) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} \left\{ \left(\sum_{i=N+1}^{\infty} \frac{F_i}{\omega_i^2-\omega^2} \right)^2 - \frac{2K}{\alpha^2-\omega^2} \times \sum_{i=N+1}^{\infty} \frac{F_i}{\omega_i^2-\omega^2} + \frac{K^2}{(\alpha^2-\omega^2)^2} \right\} d\omega. \tag{5}$$

The problem that we face, then, is to minimize $e(K,\alpha)$ subject to the constraint $\alpha > \omega_c$. Since the first part of $e(K,\alpha)$ is independent of K and α, this optimization problem is equivalent to finding K and α that minimize the following cost function.

$$\tilde{e}(K,\alpha) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} \left\{ \frac{K^2}{(\alpha^2-\omega^2)^2} - \frac{2K}{\alpha^2-\omega^2} \times \sum_{i=N+1}^{\infty} \frac{F_i}{\omega_i^2-\omega^2} \right\} d\omega. \tag{6}$$

Therefore, we intend to solve the following optimization problem.

$$\min_{K\in\mathbf{R},\ \alpha>\omega_c} \tilde{e}(K,\alpha). \tag{7}$$

It is straight-forward, but tedious to find an analytic expression for $\tilde{e}$.

$$\begin{aligned}\tilde{e}(K,\alpha) &= \frac{1}{2\pi}\Bigg\{\frac{K^2\omega_c}{\alpha^2(\alpha^2-\omega_c^2)}+\frac{K^2}{2\alpha^3}\ln\left(\frac{\alpha+\omega_c}{\alpha-\omega_c}\right)\\ &\quad -2K\sum_{i=N+1}^{\infty}\frac{F_i}{\omega_i^2-\alpha^2}\left(\frac{1}{\alpha}\ln\left(\frac{\alpha+\omega_c}{\alpha-\omega_c}\right)-\frac{1}{\omega_i}\ln\left(\frac{\omega_i+\omega_c}{\omega_i-\omega_c}\right)\right)\Bigg\}. \qquad (8)\end{aligned}$$

The optimization problem defined by (7) and (8) is a non-convex optimization problem. There are a large number of optimization methods that can be employed to find a minimum of $\tilde{e}(K,\alpha)$. However, any such minimum could be just a local minimum of $\tilde{e}$. Moreover, to use any optimization routine, we have to first truncate the infinite series term in (8). This can be done by truncating the series at a point which corresponds to a mode that is located at a frequency which is considerably higher than ω_c. This will not cause a major difficulty since as $i\to\infty$, then $\omega_i\to\infty$ too. It can be shown that in this case the very high order terms of the series will approach zero.

At this stage, we point out that if α is fixed, then $\tilde{e}$ will be a convex function of K. Therefore, for a fixed α, the optimum K can be found to be:

$$K=\frac{\sum_{i=N+1}^{\infty}\frac{F_i}{\omega_i^2-\alpha^2}\left(\frac{1}{\alpha}\ln\left(\frac{\alpha+\omega_c}{\alpha-\omega_c}\right)-\frac{1}{\omega_i}\ln\left(\frac{\omega_i+\omega_c}{\omega_i-\omega_c}\right)\right)}{\frac{\omega_c}{\alpha^2(\alpha^2-\omega_c^2)}+\frac{1}{2\alpha^3}\ln\left(\frac{\alpha+\omega_c}{\alpha-\omega_c}\right)} \qquad (9)$$

Hence, to avoid the constrained optimization problem (7), one could fix α at a frequency higher that ω_c, and then, determine K from (9). However, this approach may not lead to a low error. Therefore, in general, it is recommended that the optimization problem (7) be solved directly using the available numerical techniques.

3 Extension to multivariable systems

In many cases, it may not be possible to achieve the necessary performance by a single actuator and sensor. Should a multiple number of sensors and actuators be necessary in control of a distributed system, it is essential that the effect of higher order modes that are truncated is captured in an optimal way. In this section, we extend the procedure that was developed in the previous section to the multivariable transfer functions of reverberant plants. In the multivariable case, the transfer function matrix of the system is given by:

$$G(s)=\sum_{i=1}^{\infty}\frac{1}{s^2+\omega_i^2}H_i \qquad (10)$$

where H_i is a matrix whose dimensions are determined by the number of actuators and sensors, i.e., inputs and outputs of the system. If the system has n outputs and m inputs, then H_i's are $m\times n$ matrices. Transfer function matrix $G(s)$ has an interesting property. All of the individual transfer functions of $G(s)$ share similar poles. Moreover, if the actuator and sensors are collocated, $G(s)$ will be a square transfer function matrix whose diagonal transfer functions possess minimum-phase zeros only. However, the off-diagonal transfer functions may have non-minimum-phase zeros since they correspond to non-collocated actuators and sensors. Following [5], it can be argued that truncating this model, could seriously disturb the zeros of the diagonal transfer functions of (10). However, the effect on the off-diagonal transfer functions may be less severe. This section is aimed at extending the model correction technique of the previous section to the case of multivariable systems.

Here, we approximate $G(s)$ by a finite number of modes. It is our intention to approximate the effect of the truncated higher order modes on the low-frequency dynamics of $G(s)$ by a second order transfer function matrix as follows.

$$\hat{G}(s) = \sum_{i=1}^{N} \frac{1}{s^2+\omega_i^2} H_i + \frac{1}{s^2+\alpha^2} K \tag{11}$$

where K is a matrix of pure gains which has similar dimensions as H_i's. To be consistent with the SISO case, we will determine K and α such that this following cost function is minimized $E(K,\alpha) = \left\| W(s)(G(s) - \hat{G}(s)) \right\|_2^2$.

For a multivariable transfer function $F(s)$, $\|F(s)\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} tr\{F^*(j\omega)F(j\omega)\}d\omega$. Moreover, $tr\{M\}$ is the trace of the square matrix M. We choose $W(s)$ to be a diagonal matrix, whose diagonal elements are ideal low-pass filters, i.e., $W = diag(w_1, w_2, \dots, w_N)$ and w_i is defined by $|w_i(j\omega)| = 1$ for $-\omega_c \le w \le \omega_c$ and zero elsewhere. Now, the cost function $E(K,\alpha)$ can be re-written as:

$$\left\| W(s) \left(\sum_{i=N+1}^{\infty} \frac{1}{s^2+\omega_i^2} H_i - \frac{1}{s^2+\alpha^2} K \right) \right\|_2^2 . \tag{12}$$

This is equivalent to:

$$\begin{aligned} E(K,\alpha) &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} \Bigg\{ \sum_{i=N+1}^{\infty}\sum_{j=N+1}^{\infty} \left(\frac{1}{(\omega_i^2-\omega^2)(\omega_j^2-\omega^2)} tr\{H_i' H_j\} \right) \\ &\qquad + \frac{1}{(\alpha^2-\omega^2)^2} tr\{K'K\} - \frac{2}{\alpha^2-\omega^2} \times \sum_{i=N+1}^{\infty} \frac{1}{\omega_i^2-\omega^2} tr\{H_i' K\} \Bigg\} d\omega. \end{aligned} \tag{13}$$

The first part of the cost function is independent of K and α. Therefore, the optimization problem can be reduced to that of finding the matrix K and the resonant frequency $\alpha > \omega_c$ such that the following cost function is minimized.

$$\begin{aligned} \tilde{E}(K,\alpha) &= \frac{1}{2\pi}\Bigg\{ \left(\frac{\omega_c}{\alpha^2(\alpha^2-\omega_c^2)} + \frac{1}{2\alpha^3}\ln\left(\frac{\alpha+\omega_c}{\alpha-\omega_c}\right) \right) tr\{K'K\} \\ &\qquad - 2\sum_{i=N+1}^{\infty} \frac{1}{\omega_i^2-\alpha^2} \left(\frac{1}{\alpha}\ln\left(\frac{\alpha+\omega_c}{\alpha-\omega_c}\right) - \frac{1}{\omega_i}\ln\left(\frac{\omega_i+\omega_c}{\omega_i-\omega_c}\right) \right) tr\{H_i' K\} \Bigg\} \end{aligned} \tag{14}$$

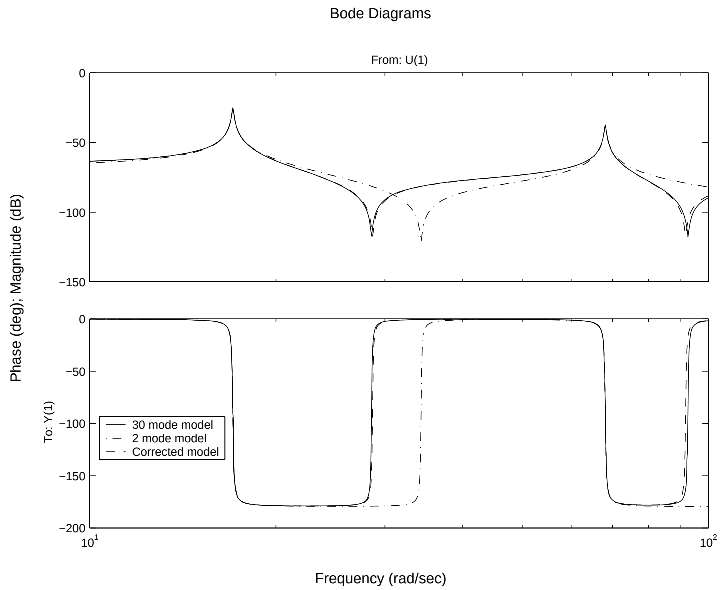

Figure 1: Comparison of the frequency responses of the thirty mode model of the beam with its two mode model and the corrected model with an out-of-bandwidth mode.

It should be clear now that the problem is reduced to solving the optimization problem $\min_{K \in \mathbf{R}^{m\times n}, \alpha > \omega_c} \tilde{E}(K, \alpha)$.

This optimization can be solved numerically. However, it does not necessarily posses a global minimum. As a result, any solution found may only be a local minimum of $\tilde{E}$. To this end, we point out that if α is fixed, the optimal K can be found by setting the derivative of $\tilde{E}$ with respect to K to zero (see page 592 0f [7]). The optimal K is found to be,

$$K = \frac{1}{\frac{\omega_c}{\alpha^2(\alpha^2-\omega_c^2)} + \frac{1}{2\alpha^3}\ln\left(\frac{\alpha+\omega_c}{\alpha-\omega_c}\right)} \sum_{i=N+1}^{\infty} \frac{1}{\omega_i^2 - \alpha^2}\left(\frac{1}{\alpha}\ln\left(\frac{\alpha+\omega_c}{\alpha-\omega_c}\right) - \frac{1}{\omega_i}\ln\left(\frac{\omega_i+\omega_c}{\omega_i-\omega_c}\right)\right) H_i. \tag{15}$$

4 Illustrative Example

In this section we apply the model correction method that has been developed in this paper to a simply supported beam with homogeneous material properties. Dimensions and physical properties of the beam are explained in [6, 8]. The beam is assumed to be 1.3 m long and it is assumed that a point force is applied to the beam at a distance of 0.075 m from one end of the beam. Here, we are concerned with the transfer function from the applied point force to the displacement at the very same point, i.e.,a colocated transfer function.

In Figure 1, we compare the truncated two mode model of the beam with its thirty mode model. Here we are allowing for a 0.3% modal damping for each mode. It can be observed that the error caused by truncation is considerably high. In the same figure, we

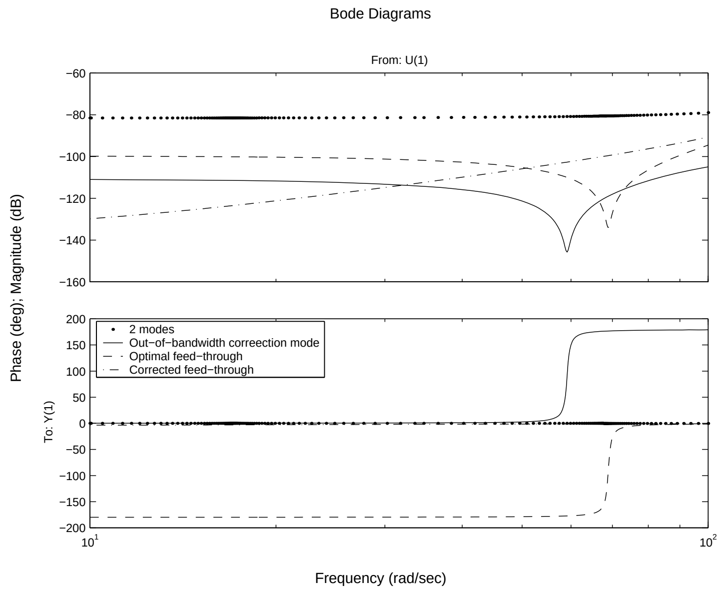

Figure 2: Comparison of the in-bandwidth error of the truncated model with three other corrected models.

have plotted the corrected two mode model which includes the correcting out-of-bandwidth mode. This term is determined using the optimization procedure developed above. The optimization is done using the constrained optimization routine of the Matlab Optimization Toolbox. The initial conditions were set at $K_0 = 6$ and $\alpha_0 = 110.8\, rad/sec$. Moreover, ω_c was chosen to be $\omega_c = (\omega_2 + \omega_3)/2$, i.e., 110.71 rad/sec. A minimum was found at $K = 2.52$ and $\alpha = 176.98\, rad/sec$.

In Figure 2, we plot the frequency responses of the error systems for the truncated model plus corrected models that are obtained by adding feed-through terms to the truncated model as suggested in [5] and [8, 6]. We also plot the error system corresponding to the corrected model that is proposed in this paper. It can be observed that all of the corrected models have lower errors than the truncated one. Moreover, it can be observed that the corrected model with the out-of-bandwidth mode results in a very small error, particularly at frequencies closer to the highest frequency of interest, i.e., ω_c.

5 Conclusions

In this paper we presented a method of minimizing the in-bandwidth error that arises in truncated assumed modes models of structures. This was done by adding a second order resonant term to the truncated model of the structure. The resonant frequency of this term was chosen to lie out of the bandwidth of interest. The resonant frequency and the gain of the correction term were determined by optimizing the $\mathcal{H}_2$ norm of the error system over the in-bandwidth frequency range. The optimization was shown to be numerically tractable. It was shown that if the resonant frequency of the correction term was fixed, an analytic expression for the optimal gain could be found.

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