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Brief Paper

Model correction for a class of spatio-temporal systems☆

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Abstract

Modal analysis has been used in modeling of a large number of physical systems such as beams, plates, acoustic enclosures, strings, etc. These models are often simplified by truncating higher frequency terms that lie out of the bandwidth of interest. Truncation can introduce a large error. This paper suggests a method of minimizing the effect of truncated modes on spatial low-frequency dynamics of the system by adding a spatial zero frequency term to the truncated model. The feed-through term is found such that the spatial $\mathscr{H}_\infty$ norm of the error system is minimized.

Keywords: Model reduction; Spatially distributed systems; Distributed parameter systems; Spatial norms; Optimization

1. Introduction

The modal analysis technique has been widely used throughout the literature to model the dynamics of spatio-temporal systems such as flexible beams and plates (Meirovitch, 1986), slewing beams (Fraser & Daniel, 1991; Book & Hastings, 1987), piezoelectric laminate beams (Alberts & Colvin, 1991) and acoustic enclosures (Hong et al., 1996). These systems have the common property that dynamics of each one of them is described by a particular partial differential equation. The modal analysis is concerned with expanding the solution of this partial differential equation in the form of an infinite series using the eigenvalues and eigenfunctions of the system.

The control designer is often only interested in devising a controller for a particular bandwidth. As a result, it is common practice to remove the modes which correspond to frequencies that lie out of the bandwidth of interest. The removed modes, however, do contribute to the low-frequency dynamics of the system. If the truncated model is then used to design a controller which is implemented on the system, say in the laboratory, the closed loop performance of the system can be considerably different from the theoretical predictions. This is mainly due to the fact that although the poles of the truncated system are at the correct frequencies, the zeros can be far away from where they should be. Therefore, it is natural to expect that a controller designed for the truncated system, may not perform well when implemented on the real system since the closed loop performance of the system is largely dictated by the open loop zeros.

This issue is addressed in Clark (1997) and a model correction technique is presented which results in a model that is closer to the real system than the truncated model. The technique of (Clark, 1997), however, applies only to SISO models and is not aimed at correcting the spatio-temporal characteristics of the system. In Moheimani (1999, 2000b) an $\mathscr{H}_2$ optimal model correction technique is proposed that applies to multivariable models of spatio-temporal systems. In Zhu and Alberts (1998), the truncation error is reduced via adding a synthetic out-of bandwidth mode to the truncated model and minimizing a similar cost function. All of these techniques are limited to correcting point-wise models of such systems. In Moheimani (2000a), this methodology is extended to allow for model correction of multi-input models of spatio-temporal systems while minimizing a spatial $\mathscr{H}_2$ norm of the error system. In this paper, we

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follow a similar procedure. However, our measure of performance is now the spatial $\mathscr{H}_\infty$ norm.

2. Modal approach to modeling

In this section, we review the mathematical basis upon which a class of spatio-temporal systems can be modeled. We consider a partial differential equation described by

$$\mathscr{L}\{y(t,r)\} + \mathscr{M}\left\{\frac{\partial^2 y(t,r)}{\partial t^2}\right\} = f(t,r). \tag{1}$$

Here, r is defined over a domain $\mathscr{R}$, $\mathscr{L}$ is a linear homogeneous differential operator of order $2p$, $\mathscr{M}$ is a linear homogeneous differential operator of order $2s$, $s \leqslant p$ and $f(t,r)$ is the system input, which could be spatially distributed over $\mathscr{R}$. Notice that $\mathscr{M}$ and $\mathscr{L}$ are spatial operators. An example is given in Section 5 that explains how these operators may be constructed for a particular system. For more examples the reader is referred to (Meirovitch, 1986). Corresponding to this partial differential equation are the following boundary conditions:

$$\mathscr{B}_\ell\{y(t,r)\} = 0, \quad \ell = 1,2,\ldots,p. \tag{2}$$

These boundary conditions are to be satisfied at every point of the boundary $\mathscr{S}$ of the domain $\mathscr{R}$. Here, $\mathscr{B}_\ell$, $\ell = 1,2,\ldots,p$ are linear differential operators of orders ranging from 0 to $2p-1$. We notice that (1) and (2) describe *spatial* and *temporal* behavior of y. It is our intention to explain how a model of y can be derived that captures the spatial and temporal characteristics of (1) and (2). The modal analysis is concerned with seeking a solution for (1) in the form

$$y(t,r) = \sum_{i=1}^{\infty} \phi_i(r) q_i(t). \tag{3}$$

Here $\phi_i(\cdot)$ are the eigenfunctions that are obtained by solving the eigenvalue problem associated with (1). That is, $\phi_i(r)$ obeys the PDE equation, $\mathscr{L}\{\phi_i(r)\} = \lambda_i \mathscr{M}\{\phi_i(r)\}$ and its associated boundary conditions, $\mathscr{B}_\ell\{\phi_i(r)\} = 0$, $\ell = 1,2,\ldots,p$, $i = 1,2,\ldots$. The solution of the eigenvalue problem consists of an infinite set of eigenvalues λ_i, $i = 1,2,\ldots$ and associated eigenfunctions $\phi_i(r)$. Assuming that the operator $\mathscr{L}$ is self-adjoint and positive definite, all the eigenvalues will be positive and can be ordered so that $\lambda_1 \leqslant \lambda_2 \leqslant \cdots$. Moreover, the eigenvalues are related to the natural frequencies of the system via $\lambda_i = \omega_i^2$, $i = 1,2,\ldots$.

In the modal analysis literature, ϕ_i's are often referred to as mode shapes. Since $\mathscr{L}$ is self-adjoint, the mode shapes possess the orthogonality property and are normalized via the following orthogonality conditions:

$$\int_{\mathscr{R}} \phi_i(r)\mathscr{L}\{\phi_j(r)\}\,\mathrm{d}r = \delta_{ij}\omega_i^2, \tag{4}$$

$$\int_{\mathscr{R}} \phi_i(r)\mathscr{M}\{\phi_j(r)\}\,\mathrm{d}r = \delta_{ij}, \tag{5}$$

where δ_{ij} is the Kronecker delta function, i.e., $\delta_{ij} = 1$ for $i = j$, and zero otherwise. To this end we point out that the expansion theorem (Meirovitch, 1986) states that series (3) will converge to the solution of (1) at every time and at every point in the domain $\mathscr{R}$ (Young, 1988). Substituting (3) in (1), we obtain

$$\mathscr{L}\left\{\sum_{i=1}^{\infty} \phi_i(r)q_i(t)\right\} + \mathscr{M}\left\{\frac{\partial^2}{\partial t^2}\sum_{i=1}^{\infty} \phi_i(r)q_i(t)\right\} = f(t,r). \tag{6}$$

Multiplying both sides of (6) by $\phi_j(r)$, integrating over the domain $\mathscr{R}$ and taking advantage of the orthogonality conditions (4) and (5), we obtain an infinite number of decoupled second order ordinary differential equations:

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = Q_i(t), \quad i = 1,2,\ldots, \tag{7}$$

where $Q_i(t) = \int_{\mathscr{R}} \phi_i(r) f(t,r)\,\mathrm{d}r$. In many cases, $Q_i(t)$ can be written as $Q_i(t) = F_i u(t)$ where $u(t)$ is the input of the system. That is, $f(t,r)$ can be decomposed into its spatial and temporal components. This is true for all physical systems of interest to us. A beam with a point force (Meirovitch, 1986), a plate with a piezoelectric actuator (Alberts & Colvin, 1991) and a flexible robotic arm (Book & Hastings, 1987) are examples of systems that satisfy this condition. Taking the Laplace transform of (7), we obtain the input–output equation of the system in terms of a transfer function:

$$G(s,r) = \sum_{i=1}^{\infty} \frac{\phi_i(r)F_i}{s^2 + \omega_i^2}. \tag{8}$$

In Section 5, we will explain how this procedure can be applied to a physical system, namely, a simply supported flexible beam.

3. Spatial norms

In the previous section we showed that modeling of spatio-temporal systems of form (1) using the modal analysis approach results in models of form (8) where r belongs to a known set, i.e., $r \in \mathscr{R}$. Moreover, the orthogonality condition (5) can often be reduced to

$$\int_{\mathscr{R}} \phi_i(r)\phi_j(r)\,\mathrm{d}r = \Phi_i^2 \delta_{ij}. \tag{9}$$

This is true for a large number of systems. For instance, beam- and plate-like structures with uniform mass distribution, acoustic enclosures with uniform cross section and uniform strings satisfy this condition. It can be observed that (8) consists of an infinite number of orthogonal modes. Moreover, G describes spatial as well as spectral behavior of the system.

In this section, we develop the mathematical machinery that is needed in proving our main results in the next section. To this end, we consider a system of the form $\mathbf{G}(s,r)$ that maps an input signal $\mathbf{w}(t) \in \mathbf{R}^m$ to an output signal $\mathbf{z}(t,r)\in\mathbf{R}^{\ell}\times\mathscr{R}$. It can be observed that the system is allowed to have a number of inputs, as well as a number of outputs each spatially distributed over the set $\mathscr{R}$. Such a system can be constructed by augmenting a number of systems, whose dynamics were studied in the previous section, to form a multi-input system. This issue will be further clarified in the sequel.

We will need the following definitions:

Definition (*Spatial $\mathscr{H}_2$ norm of a signal*). Consider a signal $\mathbf{z}(t,r)\in\mathbf{R}^{\ell}\times\mathscr{R}$. Then, the *spatial $\mathscr{H}_2$ norm* of $\mathbf{z}$ is defined as

$$\langle\langle \mathbf{z}\rangle\rangle_2^2=\int_0^{\infty}\int_{\mathscr{R}}\mathbf{z}(t,r)'\mathbf{z}(t,r)\,\mathrm{d}r\,\mathrm{d}t. \tag{10}$$

Remark. The spatial $\mathscr{H}_2$ norm of $\mathbf{z}(t,r)$ can be interpreted as the total energy of the signal $\mathbf{z}$.

Definition (*Spatial $\mathscr{H}_2$ norm of a system*). Consider a system $\mathbf{G}(s,r)$ with $r\in\mathscr{R}$. The *spatial $\mathscr{H}_2$ norm* of this system is defined as

$$\langle\langle \mathbf{G}(s,r)\rangle\rangle_2^2=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{\mathscr{R}}tr\{\mathbf{G}(\mathrm{j}\omega,r)^*\mathbf{G}(\mathrm{j}\omega,r)\}\,\mathrm{d}r\,\mathrm{d}\omega, \tag{11}$$

where $tr\{\mathbf{M}\}$ is the trace of matrix $\mathbf{M}$ and $\mathbf{M}^*$ is the complex conjugate transpose of the matrix $\mathbf{M}$.

Remark. For a single input, single output system, such as a beam with a single point force, $\langle\langle G(s,r)\rangle\rangle_2^2$ is a measure of the volume underneath the surface defined by $|G(\mathrm{j}\omega,r)|^2$. Hence, this is a natural extension of the standard $\mathscr{H}_2$ norm of linear systems to systems of form (8). Similar interpretations can be made for transfer functions of plates, etc.

Definition (*Spatial induced norm of a system*). Let $\mathscr{G}$ be the linear operator which maps the inputs of $\mathbf{G}(s,r)$ to its outputs. The *spatial induced norm of $\mathscr{G}$* is defined as

$$\langle\langle \mathscr{G}\rangle\rangle^2=\sup_{0\neq\mathbf{w}\in\mathscr{L}_{2[0,\infty]}}\frac{\langle\langle \mathbf{z}\rangle\rangle_2^2}{\|\mathbf{w}\|_2^2}. \tag{12}$$

Definition (*Spatial $\mathscr{H}_\infty$ norm of a system*). Consider a system $\mathbf{G}(s,r)$. The *spatial $\mathscr{H}_\infty$ norm* of this system is defined as

$$\langle\langle \mathbf{G}\rangle\rangle_\infty^2=\sup_{\omega\in\mathbf{R}}\lambda_{\max}\left(\int_{\mathscr{R}}\mathbf{G}(\mathrm{j}\omega,r)^*\mathbf{G}(\mathrm{j}\omega,r)\,\mathrm{d}r\right). \tag{13}$$

The following theorem proves that the spatial $\mathscr{H}_\infty$ norm of $\mathbf{G}(s,r)$ is indeed equivalent to its spatial induced norm of $\mathscr{G}$.

Theorem 1. *Suppose a stable linear system has a transfer function matrix $\mathbf{G}(s,r)$ and let $\mathscr{G}$ denote the linear map it induces from the $\mathscr{L}_2$ spaces of its inputs to its infinite-dimensional outputs. Its induced operator norm $\langle\langle \mathscr{G}\rangle\rangle$ satisfies*

$$\langle\langle \mathscr{G}\rangle\rangle=\langle\langle \mathbf{G}\rangle\rangle_\infty.$$

Proof. A proof can be found in Moheimani and Heath (2000).

4. Model correction

Consider the spatio-temporal system (8) and its corresponding orthogonality condition (9). In a typical control design scenario, the designer is often interested only in designing a controller for a particular bandwidth. Therefore, an approximate model of the system is needed that best represents the dynamics of the system in the prescribed frequency range. In other words, a lower order dynamical model is needed. A simple-minded choice in this case is simply to ignore the modes which correspond to the frequencies that lie outside the bandwidth of interest. For instance, if ω_N is equivalent or larger than the highest frequency of interest, one may choose to approximate $G(s,r)$ by $G_N(s,r)=\sum_{i=1}^N F_i\phi(r)/(s^2+\omega_i^2)$. This seems to be the mainstream approach in simplifying the dynamics of this class of systems (Clark, 1997). A drawback of this approximation is that the truncated higher order modes may contribute to the low-frequency dynamics, mainly in the form of distorting zero locations. Furthermore, these removed modes can significantly distort the spatial characteristics of the low-order model. Therefore, an approximate low-order model is needed that best captures the effect of truncated modes on the spectral (hence temporal) and spatial dynamics of the system.

To see how truncation can perturb low-frequency dynamics, we look at the error system generated by approximating the full order system with the truncated model, i.e., $\sum_{i=N+1}^{\infty}F_i\phi(r)/(s^2+\omega_i^2)$. At DC this amounts to a spatial error of $k(r)=\sum_{i=N+1}^{\infty}F_i\phi_i(r)/\omega_i^2$. This could introduce a significant amount of error. Therefore, it is sensible to expect that this error can be reduced if $k(r)$ is added to the truncated model. In the aeroelasticity literature this technique is referred to as the mode acceleration method (see Bisplinghoff & Ashley, 1962, p. 350).

To this end, we wish to generalize our approach to allow for multi-input multi-output transfer functions. We consider transfer function matrices of the form

$$\mathbf{G}(s,r)=\sum_{i=1}^{\infty}\frac{\phi_i(r)}{s^2+\omega_i^2}\mathbf{H}_i. \tag{14}$$

Here, $\mathbf{H}_i$ is a row matrix, i.e., $\mathbf{H}_i=[\mathbf{f}_1^i\;\mathbf{f}_2^i\;\dots\;\mathbf{f}_m^i]$ where m is the number of actuators, and each vector $\mathbf{f}_j^i$ is made up of l terms corresponding to the number of outputs. This requires the assumption that each output can be

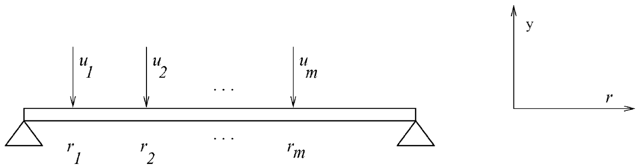

Fig. 1. A simply supported beam with m point forces.

modeled with the same set of eigenfunctions (a typical example would be where both position and acceleration are measured across $\mathscr{R}$). Moreover, the ϕ_i's satisfy the orthogonality condition (9). As a single output example, for the simply supported beam of Fig. 1 which is subject to m point forces at $r_1, \dots, r_m$, this amounts to $\mathbf{f}_s^i = \phi_i(r_s)$ for $s = 1,2,\dots,m$ where $\phi_i(r)$ and ω_i are given by (28) and (5).

A truncated version of (14) is

$$\mathbf{G}_N(s,r) = \sum_{i=1}^{N} \frac{\phi_i(r)}{s^2 + \omega_i^2}\mathbf{H}_i. \tag{15}$$

Our approach to reducing the truncation error is to add a feed-through term $\mathbf{K}(r)$ to $\mathbf{G}_N(s,r)$ such that the spatial $\mathscr{H}_\infty$ norm of the error system is minimized. Our model correction technique is based on approximating (14) with

$$\hat{\mathbf{G}}(s,r) = \mathbf{G}_N(s,r) + \mathbf{K}(r), \tag{16}$$

where $\mathbf{K}(r) = \sum_{i=N+1}^{\infty} \phi_i(r)\mathbf{K}_i$ and $\mathbf{K}_i = [\mathbf{k}_1^i \ \mathbf{k}_2^i \ \dots \ \mathbf{k}_m^i]$.

Moreover, we intend to determine the $\mathbf{K}_i$'s in a way that the following cost function is minimized:

$$J = \langle\!\langle W(s,r)(\mathbf{G}(s,r) - \hat{\mathbf{G}}(s,r)) \rangle\!\rangle_\infty^2, \tag{17}$$

where $\mathbf{G}$ and $\hat{\mathbf{G}}$ are defined as in (14) and (16), and $W(s,r)$ is an ideal low-pass weighting function distributed spatially over $\mathscr{R}$ with its cut-off frequency ω_c chosen to lie within the interval $\omega_c \in (\omega_N, \omega_{N+1})$. That is,

$$|W(j\omega, r)| = \begin{cases} 1 & -\omega_c \leqslant \omega \leqslant \omega_c, \quad r \in \mathscr{R}, \\ 0, & \text{elsewhere.} \end{cases} \tag{18}$$

We notice that $\mathbf{G}(s,r) - \hat{\mathbf{G}}(s,r)$ has no resonant poles in the frequency range of $0 \leqslant \omega \leqslant \omega_c$. Therefore, the cost function (17) will be finite.

It turns out (under some mild assumptions) that the cost function (17) is minimized by setting

$$\mathbf{K}(r) = \sum_{i=N+1}^{\infty} \phi_i(r)\mathbf{K}_i^{\text{opt}} \tag{19}$$

with

$$\mathbf{K}_i^{\text{opt}} = \frac{1}{2}\left(\frac{1}{\omega_i^2} + \frac{1}{\omega_i^2 - \omega_c^2}\right)\mathbf{H}_i. \tag{20}$$

To see this we first require three lemmas. In the first we show we can exploit orthogonality to express the cost function as the supremum over a closed frequency range of the maximum eigenvalue of a sum of error terms:

Lemma 4.1. *Define the $m \times m$ matrix*

$$\mathbf{E}_i(\omega, \mathbf{K}_i) = \Phi_i^2\left[\frac{1}{\omega_i^2 - \omega^2}\mathbf{H}_i - \mathbf{K}_i\right]'\left[\frac{1}{\omega_i^2 - \omega^2}\mathbf{H}_i - \mathbf{K}_i\right]. \tag{21}$$

Then the cost function J in (17) *can be written as*

$$J = \sup_{-\omega_c \leqslant \omega \leqslant \omega_c} \lambda_{\max}\left[\sum_{i=N+1}^{\infty} \mathbf{E}_i(\omega, \mathbf{K}_i)\right]. \tag{22}$$

Proof. Applying the definition of the spatial $\mathscr{H}_\infty$ norm (13) we may say

$$J = \sup_{\omega \in \mathbf{R}} \lambda_{\max}\left[\int_{\mathscr{R}} |W(j\omega, r)|^2 (\mathbf{G}(j\omega, r) - \hat{\mathbf{G}}(j\omega, r))^* \times (\mathbf{G}(j\omega, r) - \hat{\mathbf{G}}(j\omega, r))\,\mathrm{d}r\right].$$

From the definition of the weighting function $W(j\omega, r)$ in (18) this reduces to

$$J = \sup_{-\omega_c \leqslant \omega \leqslant \omega_c} \lambda_{\max}\left[\int_{\mathscr{R}} (\mathbf{G}(j\omega, r) - \hat{\mathbf{G}}(j\omega, r))^* \times (\mathbf{G}(j\omega, r) - \hat{\mathbf{G}}(j\omega, r))\,\mathrm{d}r\right].$$

But

$$\begin{aligned}
&\int_{\mathscr{R}} (\mathbf{G}(j\omega, r) - \hat{\mathbf{G}}(j\omega, r))^*(\mathbf{G}(j\omega, r) - \hat{\mathbf{G}}(j\omega, r))\,\mathrm{d}r \\
&= \int_{\mathscr{R}} \left(\sum_{i=N+1}^{\infty} \phi_i(r)\left[\frac{1}{\omega_i^2 - \omega^2}\mathbf{H}_i - \mathbf{K}_i\right]\right)' \\
&\quad \times \left(\sum_{k=N+1}^{\infty} \phi_k(r)\left[\frac{1}{\omega_k^2 - \omega^2}\mathbf{H}_k - \mathbf{K}_k\right]\right)\mathrm{d}r \\
&= \sum_{i=N+1}^{\infty} \Phi_i^2\left[\frac{1}{\omega_i^2 - \omega^2}\mathbf{H}_i - \mathbf{K}_i\right]'\left[\frac{1}{\omega_i^2 - \omega^2}\mathbf{H}_i - \mathbf{K}_i\right] \\
&= \sum_{i=N+1}^{\infty} \mathbf{E}_i(\omega, \mathbf{K}_i).
\end{aligned}$$

Hence the result. □

In our second lemma we consider the structure of each term $\mathbf{E}_i(\omega, \mathbf{K}_i)$. In particular we consider its structure over all frequencies $-\omega_c \leqslant \omega \leqslant \omega_c$ when we set $\mathbf{K}_i = \mathbf{K}_i^{\text{opt}}$ with $\mathbf{K}_i^{\text{opt}}$ defined by (20). We also consider its structure over all $\mathbf{K}_i$ at frequencies $\omega = 0$ and $\omega = \omega_c$. We also show that each $\mathbf{K}_i^{\text{opt}}$ minimizes the cost function

$$J_i = \sup_{-\omega_c \leqslant \omega \leqslant \omega_c} \lambda_{\max}[\mathbf{E}_i(\omega, \mathbf{K}_i)]$$

Lemma 4.2. *With* $\mathbf{E}_i(\omega, \mathbf{K}_i)$ *defined by* (21) *and for* $\mathbf{K}_i^{\text{opt}}$ *defined by* (20), *then for any column vector* $\mathbf{x}$ *and for* *any* $\mathbf{K}_i$ *we can say*

(1) $\mathbf{E}_i(0,\mathbf{K}_i^{\text{opt}}) = \mathbf{E}_i(\omega_c,\mathbf{K}_i^{\text{opt}}) = \frac{1}{4}\Phi_i^2(\mathbf{H}_i'\mathbf{H}_i)(1/\omega_i^2 - 1/(\omega_i^2-\omega_c^2))^2$,
(2) $\mathbf{x}'\mathbf{E}_i(\omega,\mathbf{K}_i^{\text{opt}})\mathbf{x} \leqslant \mathbf{x}'\mathbf{E}_i(0,\mathbf{K}_i^{\text{opt}})\mathbf{x}$ *for* $-\omega_c \leqslant \omega \leqslant \omega_c$,
(3) $\mathbf{x}'\mathbf{E}_i(0,\mathbf{K}_i^{\text{opt}})\mathbf{x} \leqslant \max[\mathbf{x}'\mathbf{E}_i(0,\mathbf{K}_i)\mathbf{x}, \mathbf{x}'\mathbf{E}_i(\omega_c,\mathbf{K}_i)\mathbf{x}]$.

Furthermore,

(4) *each term* $\mathbf{K}_i^{\text{opt}}$ *defined by* (20) *satisfies* $\mathbf{K}_i^{\text{opt}} = \arg\inf_{\mathbf{K}_i} \max_{-\omega_c\leqslant\omega\leqslant\omega_c} \lambda_{\max}[\mathbf{E}_i(\omega,\mathbf{K}_i)]$.

Proof. A proof is included in the appendix. □

In our third lemma we consider summing terms. Define $\mathscr{K}_M$ as the set

$$\mathscr{K}_M = \{\mathbf{K}_{N+1}, \mathbf{K}_{N+2}, \ldots, \mathbf{K}_{N+M}\}$$

with $\mathscr{K}_M^{\text{opt}} = \{\mathbf{K}_{N+1}^{\text{opt}}, \mathbf{K}_{N+2}^{\text{opt}}, \ldots, \mathbf{K}_{N+M}^{\text{opt}}\}$. Define also $\mathbf{S}_M(\omega,\mathscr{K}_M) = \sum_{i=N+1}^{N+M}\mathbf{E}_i(\omega,\mathbf{K}_i)$ and the cost

$$\bar{J}_M(\mathscr{K}_M) = \sup_{-\omega_c\leqslant\omega\leqslant\omega_c} \lambda_{\max}[\mathbf{S}_M(\omega,\mathscr{K}_M)]. \tag{23}$$

We show that the cost $\bar{J}_M(\mathscr{K}_M)$ is optimised at $\mathscr{K}_M^{\text{opt}}$. In particular we show that $\mathbf{S}_M(\omega,\mathscr{K}_M)$ has the same properties we showed for each term $\mathbf{E}_i(\omega,\mathbf{K}_i)$ in Lemma 4.2.

Lemma 4.3. *For all* M, *for any column vector* $\mathbf{x}$ *and for any* $\mathscr{K}_M$ *we can say*

(1) $\mathbf{S}_M(0,\mathscr{K}_M^{\text{opt}}) = \mathbf{S}_M(\omega_c,\mathscr{K}_M^{\text{opt}}) = \frac{1}{4}\sum_{i=N+1}^{N+M}\Phi_i^2(\mathbf{H}_i'\mathbf{H}_i)(1/\omega_i^2 - 1/(\omega_i^2-\omega_c^2))^2$,
(2) $\mathbf{x}'\mathbf{S}_M(\omega,\mathscr{K}_M^{\text{opt}})\mathbf{x} \leqslant \mathbf{x}'\mathbf{S}_M(0,\mathscr{K}_M^{\text{opt}})\mathbf{x}$ *for* $-\omega_c \leqslant \omega \leqslant \omega_c$,
(3) $\mathbf{x}'\mathbf{S}_M(0,\mathscr{K}_M^{\text{opt}})\mathbf{x} \leqslant \max[\mathbf{x}'\mathbf{S}_M(0,\mathscr{K}_M)\mathbf{x}, \mathbf{x}'\mathbf{S}_M(\omega_c,\mathscr{K}_M)\mathbf{x}]$.

Furthermore,

(4) *the cost* $\bar{J}_M(\mathscr{K}_M)$ *is minimised at* $\mathscr{K}_M = \mathscr{K}_M^{\text{opt}}$.

Proof.

(1) This follows immediately from Lemma 4.2.
(2) By induction:
Suppose the result is true for M. Then by supposition and from Lemma 4.2

$$\begin{aligned}\mathbf{x}'\mathbf{S}_{M+1}(\omega,\mathscr{K}_{M+1}^{\text{opt}})\mathbf{x} &= \mathbf{x}'\mathbf{S}_M(\omega,\mathscr{K}_M^{\text{opt}})\mathbf{x} \\ &\quad + \mathbf{x}'\mathbf{E}_{N+M+1}(\omega,\mathbf{K}_{N+M+1}^{\text{opt}})\mathbf{x} \\ &\leqslant \mathbf{x}'\mathbf{S}_M(0,\mathscr{K}_M^{\text{opt}})\mathbf{x} \\ &\quad + \mathbf{x}'\mathbf{E}_{N+M+1}(0,\mathbf{K}_{N+M+1}^{\text{opt}})\mathbf{x} \\ &= \mathbf{x}'\mathbf{S}_{M+1}(0,\mathscr{K}_{M+1}^{\text{opt}})\mathbf{x}.\end{aligned}$$

But we also know from Lemma 4.2 that the result is true for $M = 1$.
(3) Following the same reasoning as for the proof of part (3) of Lemma 4.2 we find that if both

$$\mathbf{x}'\mathbf{S}_M(0,\mathscr{K}_M)\mathbf{x} < \mathbf{x}'\mathbf{E}_i(0,\mathscr{K}_M^{\text{opt}})\mathbf{x}$$

and

$$\mathbf{x}'\mathbf{S}_M(\omega_c,\mathscr{K}_M)\mathbf{x} < \mathbf{x}'\mathbf{S}_M(0,\mathscr{K}_M^{\text{opt}})\mathbf{x}$$

then

$$\sum_{i=N+1}^{N+M} 2\Phi_i^2\left|\frac{1}{2}\left(\frac{1}{\omega_i^2} + \frac{1}{\omega_i^2+\omega_c^2}\right)\mathbf{H}_i\mathbf{x} - \mathbf{K}_i\mathbf{x}\right|^2 < 0$$

which cannot be true.
(4) This is straightforward, following the same reasoning as part (4) of Lemma 4.2. □

We can now state our main result, which follows immediately:

Theorem 2. *Assume we have* $\lim_{M\to\infty}(J - \bar{J}_M) = 0$ *with* $\bar{J}_M$ *defined in* (23) *and evaluated at any* $\mathscr{K}_M = \{\mathbf{K}_{N+1}, \mathbf{K}_{N+2}, \ldots, \mathbf{K}_{N+M}\}$, *and* J *defined in* (17) *and evaluated at* $\mathbf{K}(r) = \sum_{i=N+1}^{N+M}\phi_i(r)\mathbf{K}_i$. *Also assume that*

$$\lim_{M\to\infty}\left\langle\!\!\left\langle \sum_{i=N+M+1}^{\infty} \phi_i(r)\mathbf{K}_i^{\text{opt}} \right\rangle\!\!\right\rangle_\infty^2 = 0.$$

Then the cost J *in* (17) *is minimized by taking* $\mathbf{K}(r) = \sum_{i=N+1}^{\infty}\phi_i(r)\mathbf{K}_i^{\text{opt}}$.

Remark. The assumptions in Theorem 2 are mild. For example, a sufficient condition for the first assumption is that for any $\mathscr{K}_M$

$$\lim_{M\to\infty}\left\langle\!\!\left\langle W(s,r)\sum_{i=N+M+1}^{\infty}\frac{\phi_i(r)}{s^2+\omega_i^2}\mathbf{H}_i\right\rangle\!\!\right\rangle_\infty^2 = 0.$$

We find

$$\begin{aligned}&\left\langle\!\!\left\langle W(s,r)\sum_{i=N+M+1}^{\infty}\frac{\phi_i(r)}{s^2+\omega_i^2}\mathbf{H}_i\right\rangle\!\!\right\rangle_\infty^2 \\ &= \sup_{-\omega_c\leqslant\omega\leqslant\omega_c}\lambda_{\max}\left\{\int_{\mathscr{R}}\left(\sum_{i=N+M+1}^{\infty}\frac{\phi_i(r)}{\omega_i^2-\omega^2}\mathbf{H}_i\right)' \times\left(\sum_{j=N+M+1}^{\infty}\frac{\phi_j(r)}{\omega_j^2-\omega^2}\mathbf{H}_j\right)\mathrm{d}r\right\} \\ &= \sup_{-\omega_c\leqslant\omega\leqslant\omega_c}\lambda_{\max}\left\{\sum_{i=N+M+1}^{\infty}\Phi_i^2\left(\frac{1}{\omega_i^2-\omega^2}\right)^2\mathbf{H}_i'\mathbf{H}_i\right\} \\ &\leqslant \left(\frac{1}{\omega_i^2-\omega_c^2}\right)^2\lambda_{\max}\left\{\sum_{i=N+M+1}^{\infty}\Phi_i^2\mathbf{H}_i'\mathbf{H}_i\right\}.\end{aligned}$$

Thus it is sufficient for the first assumption either that the $\mathbf{H}_i$'s are bounded and there is a finite minimum separation between all resonant frequencies ω_i (above an arbitrarily chosen high bandwidth), or that the $\mathbf{H}_i$'s tend to zero sufficiently fast. It is straightforward to show that such conditions are also sufficient for our second assumption to be valid.

These conditions are easy to check on a case-by-case basis for such applications as flexible beams and plates and acoustic ducts and enclosures. These systems are of particular interest to us. In this context, we note (Hughes, 1987) that the utmost high-order modes are an artificial construct of the modelling process; we should expect them to have a vanishing effect on the low-frequency behavior.

Observation. A direct implication of Theorem (2) is that if our model correction technique is applied to each individual transfer function, the resulting corrected multi-input system will be optimal in the sense of (17). This means that the result can be applied to each transfer function term by term and similarly to each mode on a term by term basis.

5. Illustrative example

In this section, we apply the model correction method which was developed in the previous section to a simply supported single-input single-output beam model. Dynamics of this system satisfies (1) and (2). Hence, modal analysis can be employed to obtain a model of this system.

Consider a simply supported beam as depicted in Fig. 2. Here, $y(t,r)$ denotes the elastic deformation of the beam as measured from the rest position. The elastic deflection $y(t,r)$ is governed by the classical Bernoulli–Euler beam equation (Meirovitch, 1986)

$$\frac{\partial^2}{\partial r^2}\left[EI\,\frac{\partial^2 y(t,r)}{\partial r^2}\right] + \rho A\,\frac{\partial^2 y(t,r)}{\partial t^2} = u(t)\delta(r-r_1), \tag{24}$$

where E, I, A, $u(t)\delta(r-r_1)$ and ρ represent Young's modulus, moment of inertia, cross-section area, external force applied at r_1, and the linear mass density of the beam, respectively.

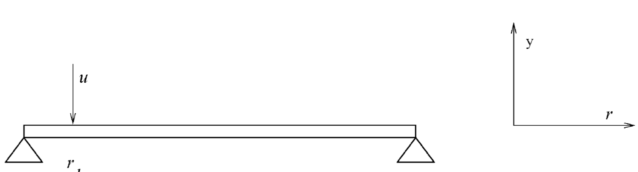

Fig. 2. A simply supported flexible beam.

Pinned–pinned beam boundary conditions are

$$y(t,0)=0, \qquad y(t,L)=0,$$
$$EI\frac{\partial^2 y(t,r)}{\partial r^2}\bigg|_{r=0}=0, \qquad EI\frac{\partial^2 y(t,r)}{\partial r^2}\bigg|_{r=L}=0. \tag{25}$$

The first two boundary conditions state that there are no movements at the two ends of the beam and the second two conditions state that the beam curvatures at both ends are zero.

Comparing (24) and (25) with (1) and(2), we notice that

$$\mathscr{L}=\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(EI\,\frac{\mathrm{d}^2}{\mathrm{d}r^2}\right), \quad \mathscr{M}=\rho A,$$
$$\mathscr{B}_1=1, \quad \mathscr{B}_2=EI\,\frac{\mathrm{d}^2}{\mathrm{d}r^2}, \quad f(t,r)=u(t)\delta(r-r_1).$$

Assuming a solution of form (3) and following the procedure that was explained earlier, we can find a transfer function of form (8) for this system. The eigenfunctions are chosen to be orthogonal according to the condition

$$\int_0^L \phi_i(r)\phi_j(r)\rho A\,\mathrm{d}r=\delta_{ij}. \tag{26}$$

The transfer function between applied force $U(s)$ and the elastic deflection of the beam $\hat{y}(s,r)$ is given by (Krishnan & Vidyasagar, 1987)

$$\frac{\hat{y}(s,r)}{U(s)}=\sum_{i=1}^{\infty}\frac{\phi_i(r_1)\phi_i(r)}{(s^2+\omega_i^2)}. \tag{27}$$

For the pinned–pinned beam system in Fig. 2 the mode functions are given by (Meirovitch, 1986)

$$\phi_i(r)=\sqrt{\frac{2}{\rho AL}}\sin\left(\frac{\mathrm{i}\pi r}{L}\right) \tag{28}$$

and the corresponding natural frequencies are $\omega_i=(\mathrm{i}\pi/L)^2\sqrt{EI/\rho A}$. The parameters of the beam are: L = beam length = 1.3 m; r_1 = 0.05 m; ρA = 0.6265 kg /m; EI = 5.329 N m^2. Moreover, in our simulations we allow for a damping ratio of 0.3% for all the modes.

In this example, we are interested in demonstrating the effectiveness of our model correction methodology applied to the truncated model of the beam. In Fig. 3, we compare frequency responses of the 30 mode model of the beam with that of the truncated 2 mode model and the corrected 2 mode model by adding the optimal feed-through term as explained in previous sections. It is assumed that the first 30 modes of the beam represent the dynamics reasonably accurately within the bandwidth of interest. The plots correspond to four distinct points along the beam. It can be observed that as a result of the truncation, the zeros are considerably displaced. However, after the optimal feed-through term is added to the truncated 2 mode model, the zeros are moved much closer to their expected locations.

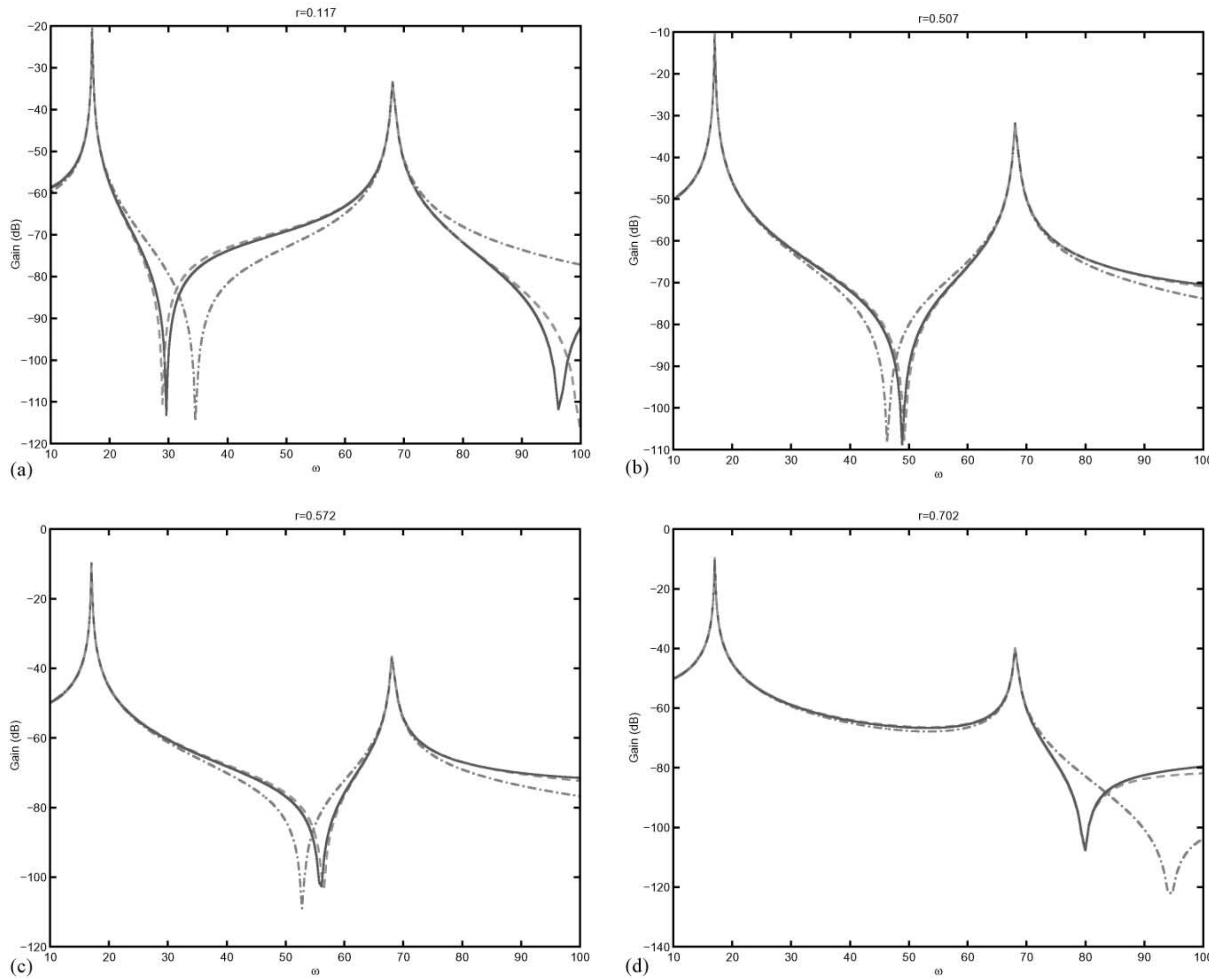

Fig. 3. Comparison of point-wise frequency responses of the beam at four distinct points, '—' 30 mode model, '- . -' 2 mode model, '- -' corrected 2 mode model.

6. Conclusion

In this paper, we introduced a class of spatio-temporal systems and their corresponding model correction problem. We explained that in the modal analysis approach, the solution of the partial differential equation that describes the dynamics of the system is expanded in the form of an infinite series. For controller design purposes this series is approximated by a finite-dimensional system via truncation. We explained that the truncation could introduce a significant error. We proposed that this error could be reduced if a feed-through term is added to the model of the system. Moreover, we demonstrated how this term could be found such that spatial and temporal characteristics of the system are best preserved within the bandwidth of interest as measured by a spatial $\mathscr{H}_\infty$ norm. We demonstrated that a spatially distributed feed-through term can capture the effect of truncated modes, and showed how this can be constructed on a term by term basis. Finally, we showed how this approach would apply to a physical system.

Appendix

Proof of Lemma 4.2.

(1) This follows from simple calculation.
(2) We can evaluate

$$\mathbf{E}_i(\omega, \mathbf{K}_i^{\text{opt}}) = \frac{1}{4}\Phi_i^2(\mathbf{H}_i'\mathbf{H}_i)\left(\frac{2}{\omega_i^2 - \omega^2} - \frac{1}{\omega_i^2} - \frac{1}{\omega_i^2 - \omega_c^2}\right)^2.$$

So

$$\mathbf{E}_i(\omega,\mathbf{K}_i^{\mathrm{opt}})-\mathbf{E}_i(0,\mathbf{K}_i^{\mathrm{opt}})$$
$$=\frac{1}{4}\Phi_i^2(\mathbf{H}_i'\mathbf{H}_i)\left[\left(\frac{2}{\omega_i^2-\omega^2}-\frac{1}{\omega_i^2}-\frac{1}{\omega_i^2-\omega_c^2}\right)^2-\left(\frac{1}{\omega_i^2}-\frac{1}{\omega_i^2-\omega_c^2}\right)^2\right]$$
$$=\Phi_i^2(\mathbf{H}_i'\mathbf{H}_i)\left(\frac{1}{\omega_i^2-\omega^2}-\frac{1}{\omega_i^2}\right)\left(\frac{1}{\omega_i^2-\omega^2}-\frac{1}{\omega_i^2-\omega_c^2}\right).$$

Hence

$$\mathbf{x}'\mathbf{E}_i(\omega,\mathbf{K}_i^{\mathrm{opt}})\mathbf{x}-\mathbf{x}'\mathbf{E}_i(0,\mathbf{K}_i^{\mathrm{opt}})\mathbf{x}$$
$$=\Phi_i^2|\mathbf{H}_i\mathbf{x}|^2\left(\frac{1}{\omega_i^2-\omega^2}-\frac{1}{\omega_i^2}\right)\left(\frac{1}{\omega_i^2-\omega^2}-\frac{1}{\omega_i^2-\omega_c^2}\right)$$
$$\leqslant 0$$

since the terms $(1/(\omega_i^2-\omega^2)-1/\omega_i^2)$ and $(1/(\omega_i^2-\omega^2)-1/(\omega_i^2-\omega_c^2))$ have opposite signs.

(3) By *reductio ad absurdum*:
Suppose we have both

$$\mathbf{x}'\mathbf{E}_i(0,\mathbf{K}_i)\mathbf{x}<\mathbf{x}'\mathbf{E}_i(0,\mathbf{K}_i^{\mathrm{opt}})\mathbf{x}$$

and

$$\mathbf{x}'\mathbf{E}_i(\omega_c,\mathbf{K}_i)\mathbf{x}<\mathbf{x}'\mathbf{E}_i(\omega_c,\mathbf{K}_i^{\mathrm{opt}})\mathbf{x}.$$

Evaluating the expressions we find both

$$\Phi_i^2\left|\frac{1}{\omega_i^2}\mathbf{H}_i\mathbf{x}-\mathbf{K}_i\mathbf{x}\right|^2-\Phi_i^2\left|\frac{1}{\omega_i^2}\mathbf{H}_i\mathbf{x}-\mathbf{K}_i^{\mathrm{opt}}\mathbf{x}\right|^2<0$$

and

$$\Phi_i^2\left|\frac{1}{\omega_i^2-\omega_c^2}\mathbf{H}_i\mathbf{x}-\mathbf{K}_i\mathbf{x}\right|^2-\Phi_i^2\left|\frac{1}{\omega_i^2-\omega_c^2}\mathbf{H}_i\mathbf{x}-\mathbf{K}_i^{\mathrm{opt}}\mathbf{x}\right|^2<0$$

and hence both

$$\Phi_i^2\mathbf{x}'\left(\frac{2}{\omega_i^2}\mathbf{H}_i-\mathbf{K}_i-\mathbf{K}_i^{\mathrm{opt}}\right)'(\mathbf{K}_i^{\mathrm{opt}}-\mathbf{K}_i)\mathbf{x}<0$$

and

$$\Phi_i^2\mathbf{x}'\left(\frac{2}{\omega_i^2+\omega_c^2}\mathbf{H}_i-\mathbf{K}_i-\mathbf{K}_i^{\mathrm{opt}}\right)'(\mathbf{K}_i^{\mathrm{opt}}-\mathbf{K}_i)\mathbf{x}<0.$$

Adding we find

$$2\Phi_i^2\mathbf{x}'\left(\left[\frac{1}{\omega_i^2}+\frac{1}{\omega_i^2+\omega_c^2}\right]\mathbf{H}_i-\mathbf{K}_i-\mathbf{K}_i^{\mathrm{opt}}\right)'\times(\mathbf{K}_i^{\mathrm{opt}}-\mathbf{K}_i)\mathbf{x}<0.$$

Substituting for K_i^{opt} from (20) gives

$$2\Phi_i^2\left|\frac{1}{2}\left(\frac{1}{\omega_i^2}+\frac{1}{\omega_i^2+\omega_c^2}\right)\mathbf{H}_i\mathbf{x}-\mathbf{K}_i\mathbf{x}\right|^2<0$$

which cannot be true.

(4) The maximum eigenvalue is given by

$$\lambda_{\max}[\mathbf{E}_i(\omega,\mathbf{K}_i^{\mathrm{opt}})]=\max_{\mathbf{x}\neq 0}\frac{\mathbf{x}'\mathbf{E}_i(\omega,\mathbf{K}_i^{\mathrm{opt}})\mathbf{x}}{\mathbf{x}'\mathbf{x}}.$$

From part (2) of Lemma 4.2 we see that

$$\lambda_{\max}[\mathbf{E}_i(\omega,\mathbf{K}_i^{\mathrm{opt}})]\leqslant\lambda_{\max}[\mathbf{E}_i(0,\mathbf{K}_i^{\mathrm{opt}})].$$

But similarly from part (3), for any $\mathbf{K}_i$,

$$\lambda_{\max}[\mathbf{E}_i(0,\mathbf{K}_i^{\mathrm{opt}})]\leqslant\max[\lambda_{\max}[\mathbf{E}_i(0,\mathbf{K}_i)],\lambda_{\max}[\mathbf{E}_i(\omega_c,\mathbf{K}_i)]]$$
$$\leqslant\max_{-\omega_c\leqslant\omega\leqslant\omega_c}\lambda_{\max}[\mathbf{E}_i(\omega,\mathbf{K}_i)].$$

Hence

$$\max_{-\omega_c\leqslant\omega\leqslant\omega_c}\lambda_{\max}[\mathbf{E}_i(\omega,\mathbf{K}_i^{\mathrm{opt}})]$$
$$\leqslant\max_{-\omega_c\leqslant\omega\leqslant\omega_c}\lambda_{\max}[\mathbf{E}_i(\omega,\mathbf{K}_i)]\quad\text{for all }\mathbf{K}_i.\qquad\square$$

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