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Brief Paper

Combining switching, over-saturation and scaling to optimise control performance in the presence of model uncertainty and input saturation☆

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Abstract

Saturating actuators are present in all real control systems. Their effect on system performance clearly depends on the range of control action required relative to the saturation bounds. Much of the prior work on this topic has centred on how to switch linear controllers so as to avoid saturation occurring. This, however, has meant that the full input authority has not been exploited in the control law. Recently, two alternative methods have been proposed for switching linear controllers so as to force the input into saturation. They achieve this goal by scaling the controls or by allowing over-saturation in the switching scheme. In this paper the two methods are combined into a more general scheme. It is also shown that the combined scheme is capable of achieving superior performance. A robust version of the algorithm is also described which is applicable to a class of uncertain systems.

Keywords: Switching control; Saturating actuators; Over-saturation; Scaling; Robust control; Uncertainties

1. Introduction

The phenomenon of input saturation is one of the more common non-linearities encountered in control system applications. The presence of saturation imposes fundamental limits on the achievable performance. In some situations, the demands on the control amplitude are such that saturation is never encountered. However, there are other cases where the performance demands are such that the input needs be pushed to the available limits so as to make best use of the available control authority. In this context, there has been recent interest in the use of logic-based switching controllers for dealing with linear systems with input saturation. Systems composed of logic-based controllers, together with the processes they are intended to control, are concrete examples of hybrid systems. A number of analytical studies of hybrid systems has emerged in the last decade. For example, a tutorial on hybrid systems stability can be found in Branicky (1997).

Methods that employ controller switching to deal with input constraints have been proposed in, for example, Tan (1992), Wredenhagen and Bélanger (1994) and Kolmanovsky and Gilbert (1996). A common feature of these controllers is that the saturation levels are avoided. Thus, the resulting controllers are, relatively, 'low gain' controllers. Recently, it has been independently recognised in Lin, Pachter, Banda, and Shamash (1997) and in De Doná, Moheimani, Goodwin, and Feuer (1999) that this strategy is conservative and that the performance of these kinds of systems can be improved by forcing the controls into saturation. Both, Lin et al. (1997) and De Doná et al. (1999), constitute alternative methods that modify the basic scheme of Wredenhagen and Bélanger (1994) in order to achieve this goal. They do so by, respectively, *scaling* the controller gains and by allowing *over-saturation*.

In this paper we examine each of the above designs and show that they can be combined into a more general scheme. We show that a judicious combination of the core ideas in the above schemes (i.e., *switching*, *over-saturation* and *scaling*) provides superior performance compared to the case where each is used separately.

2. Piecewise-linear LQ control

Several contributions reported in the literature (Tan, 1992; Wredenhagen & Bélanger, 1994; Ledwich, 1995;

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Kolmanovsky & Gilbert, 1996; Lin et al., 1997; De Doná et al., 1999) indicate that, in the presence of input constraints, it is beneficial to have a number of different precomputed gains $K_1, K_2, \ldots, K_N$, in a sequence related to improved performance and to select, via a switching strategy, the more appropriate one according to the operating condition. A seminal idea in the design of switching controllers having guaranteed stability, is the piecewise-linear LQ control (PLC) algorithm which appeared originally in Wredenhagen and Bélanger (1994). The PLC control strategy is based on LQ theory and the associated switching surfaces are positively invariant sets given by nested ellipsoids (a non-empty set $\mathscr{E} \subset \mathbb{R}^n$ is positively invariant if for a dynamical system and for any initial condition $x(t_0) \in \mathscr{E}$, then $x(t) \in \mathscr{E}$ for all $t \geqslant t_0$). A key idea in the PLC controller is that, in each switching region, a linear controller is selected such that the constraints imposed by saturation are not violated. In the sequel, we will briefly expand on the main ideas used in the PLC controller approach.

Consider a general nth order linear dynamical system subject to input saturation

$$\dot{x}(t) = Ax(t) + B\,\mathrm{sat}_{\tilde{\Delta}}(u(t)), \quad x(t_0) = x_0 \in \mathscr{X} \subset \mathbb{R}^n, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, and the pair (A, B) is assumed to be stabilisable. We assume full state measurement, and the input saturation function $\mathrm{sat}_{\tilde{\Delta}} : \mathbb{R}^m \to \mathbb{R}^m$, for the vector of saturation bounds $\tilde{\Delta} = (\Delta_1, \ldots, \Delta_m)$, is defined by

$$\mathrm{sat}_{\tilde{\Delta}}(u) \triangleq [\mathrm{sat}_{\Delta_1}(u_1), \mathrm{sat}_{\Delta_2}(u_2), \ldots, \mathrm{sat}_{\Delta_m}(u_m)]^{\mathrm{T}}, \tag{2}$$

where each $\mathrm{sat}_{\Delta_i}(\cdot)$ function is defined by $\mathrm{sat}_{\Delta_i}(u_i) \triangleq \mathrm{sgn}(u_i) \times \min\{|u_i|, \Delta_i\}$.

The PLC controller design starts with a sequence $\{\rho_i\}_{i=1}^N$ of N design parameters such that $\rho_1 > \rho_2 > \cdots > \rho_N > 0$, and an $n \times n$ design matrix $Q \geqslant 0$. Then, for each ρ_i and matrix Q, we form a diagonal matrix $R_i = \mathrm{diag}(r_i^1, r_i^2, \ldots, r_i^m)$, where $r_i^j > 0$, $j = 1, 2, \ldots, m$. The choices of the sequence $\{\rho_i\}_{i=1}^N$ and the construction of the corresponding diagonal matrices R_i are explained later. For each R_i, compute P_i and K_i such that

$$0 = P_i A + A^{\mathrm{T}} P_i - P_i B R_i^{-1} B^{\mathrm{T}} P_i + Q, \tag{3}$$

$$K_i = R_i^{-1} B^{\mathrm{T}} P_i, \tag{4}$$

where P_i is the positive definite solution of the algebraic Riccati equation (3) for the optimal LQ problem and the gain K_i is such that $u(t) = -K_i x(t)$ minimises the cost

$$J(x_0) = \int_{t_0}^{+\infty} [x^{\mathrm{T}}(t) Q x(t) + u^{\mathrm{T}}(t) R_i u(t)]\,\mathrm{d}t \tag{5}$$

when $x(t)$ satisfies (1) without saturation.

The switching surfaces are ellipsoids defined by

$$\mathscr{E}_i = \mathscr{E}_i(P_i, \rho_i) \triangleq \{x : x^{\mathrm{T}} P_i x \leqslant \rho_i\}, \tag{6}$$

which can be shown, by simple Lyapunov analysis, to be positively invariant sets for system (1) under the control $u(t) = -K_i x(t)$, where P_i and K_i are solutions of (3) and (4) for a given R_i. In Corollary 3.1 below, we will present a proof of the invariance of the ellipsoids $\mathscr{E}_i$ in a more general setting. The elements of $R_i = \mathrm{diag}(r_i^1, r_i^2, \ldots, r_i^m)$ are chosen, for a given ρ_i, to be the largest such that

$$|u_j| = \left| \frac{1}{r_i^j} b_j^{\mathrm{T}} P_i x \right| \leqslant (1 + \bar{\beta}_j)\Delta_j, \quad \forall x \in \mathscr{E}_i(P_i, \rho_i), \tag{7}$$

$j = 1, 2, \ldots, m$, where u_j is the jth element of u, and b_j is the jth column of matrix B. The constants $\bar{\beta}_j$, $j = 1, 2, \ldots, m$ are included here for later reference, but it is *important* to realise that, in the PLC controller, they are: $\bar{\beta}_j = 0$, $j = 1, 2, \ldots, m$. The existence and uniqueness of such $R_i = \mathrm{diag}(r_i^1, r_i^2, \ldots, r_i^m)$ are established in Wredenhagen and Bélanger (1994), and an iterative algorithm for their computation is provided, together with a proof of its convergence.

The procedure used to choose the design parameters ρ_i, $i = 1, 2, \ldots, N$, is as follows: Given a set of initial conditions $\mathscr{X} \subset \mathbb{R}^n$, choose $\rho_1 > 0$ such that $\mathscr{X} \subset \mathscr{E}_1(P_1, \rho_1)$, where P_1 is computed from (3) with R_1 such that (7) is satisfied (with, in the case of the PLC controller, $\bar{\beta}_j = 0$, $j = 1, \ldots, m$). The rest of the ρ_i are chosen as successively smaller real numbers such that $\rho_1 > \rho_2 > \cdots > \rho_N > 0$. As pointed out in Wredenhagen and Bélanger (1994), the choice of the parameters ρ_i and the number of ellipsoids N requires some experimentation and the exact location of each of the $\mathscr{E}_i$, in order to optimise the performance, remains an open question.

In Wredenhagen and Bélanger (1994) it is also proven that the ellipsoids in the sequence $\{\mathscr{E}_i\}_{i=1}^N$ are nested, i.e. $\mathscr{E}_{i+1} \subset \mathscr{E}_i$, for each $i = 1, 2, \ldots, N-1$. This nesting property allows us to perform a partitioning of the state space region contained into the biggest ellipsoid in N cells: $\{\mathscr{C}_i\}_{i=1}^N$ defined as: $\mathscr{C}_i = \mathscr{E}_i \backslash \mathscr{E}_{i+1}$, for $i = 1, 2, \ldots, N-1$, and $\mathscr{C}_N = \mathscr{E}_N$. The PLC controller is then defined by the switching strategy:

$$u = -K_i x \quad \text{for } x \in \mathscr{C}_i, \; i = 1, 2, \ldots, N. \tag{8}$$

As can be deduced from (8), the controller does not cover the whole state space $\mathbb{R}^n$. However, for stable and quasi-stable open loop plants, it is possible to cover all of $\mathbb{R}^n$ whilst maintaining closed loop stability by letting $\rho_1 \to \infty$ (Wredenhagen & Bélanger, 1994). For unstable plants, the ellipsoids tend to a limiting ellipsoid and hence it is impossible to cover all of $\mathbb{R}^n$. The proof that system (1) with PLC control (8) is asymptotically stable for all $x \in \mathscr{E}_1$ is a direct consequence of the ellipsoids $\mathscr{E}_i$ being positively invariant. Since this proof is contained as a special case of Theorem 3.2 below, it will not be presented here. (See Remark 3.2.)

3. PLC control with allowed over-saturation

A distinctive feature of the PLC controller reviewed in the previous section is that, in order to ensure that the input constraints are avoided, the scheme yields a relatively 'low gain' controller. In fact, this method is conservative since, by taking $\bar{\beta}_j = 0$, $j = 1, 2, \ldots, m$ in (7), the control is away from the saturation level 'almost everywhere'. Here we present a switching strategy based on the PLC design which allows some prescribed level of over-saturation $\bar{\beta}_j > 0$, thus providing better utilisation of the full power of the available control authority. By the term 'over-saturation', we mean a situation in which a controller initially demands an input level greater than the available range followed by truncation via a simple saturation operation. In this section we concentrate on providing stability results for the resulting PLC control including allowed over-saturation. Simulation examples, presented later, show that it is always beneficial, in terms of improved performance, to allow for some level of over-saturation. In order to measure the magnitude of control saturation, an over-saturation index is defined as follows.

Definition 3.1. Given a saturation function $\mathrm{sat}_{\Delta_i}(\cdot)$ and a scalar control signal $u_i(t)$ we define a function $\beta_i(t)$ as

$$\beta_i(t) = \begin{cases} \dfrac{u_i(t) - \mathrm{sat}_{\Delta_i}(u_i(t))}{\mathrm{sat}_{\Delta_i}(u_i(t))} & \text{for } u_i(t) \neq 0, \\ 0 & \text{for } u_i(t) = 0. \end{cases} \tag{9}$$

Clearly, $\beta_i(t) = 0$ whenever $u_i(t)$ is not saturated, and $\beta_i(t)$ is the relative value of the demanded control with respect to the saturation bound Δ_i when $u_i(t)$ is saturated. The over-saturation index is then defined as a constant $\bar{\beta}_i$ such that the allowed supremum of the control signal is $\|\beta_i(t)\|_\infty \leqslant \bar{\beta}_i$.

The design of the over-saturated law follows as for the PLC controller explained in Section 2, with the only difference being that, in (7), some level of over-saturation is allowed. Specifically, the elements of $R_i = \mathrm{diag}(r_i^1, r_i^2, \ldots, r_i^m)$ are chosen, for a given ρ_i, to be the largest values such that (7) is satisfied with over-saturation index $\bar{\beta}_j > 0$ for the control u_j and saturation bound Δ_j, as defined in Definition 3.1. Notice that the existence and uniqueness of such a sequence $R_i = \mathrm{diag}(r_i^1, r_i^2, \ldots, r_i^m)$ can, again, be established using the results of Wredenhagen and Bélanger (1994) since the control constraints are arbitrary. Thus, the same algorithms can be used for computing R_i as in standard PLC control.

Given system (1), we assume:

Assumption 3.1. The pair (A, B) is stabilizable.

Assumption 3.2. The design matrix Q in (3) is positive definite, denoted as $Q > 0$.

Assumption 3.1 is a necessary condition for the existence of a unique positive definite solution to the ARE (3). Assumption 3.2 is a design choice that will be relaxed later. We prove, in the following theorem, that the stability of the PLC switching scheme is retained when some level of over-saturation is allowed; namely, we allow over-saturation, for each element u_j of the control vector u, up to

$$[\bar{\beta}_{\max}]_j = \min_{i=1,\ldots,N} \sqrt{1 + \frac{4\lambda_{\min}(Q)}{(\sum_{l=1}^m r_i^l) k_i^j (k_i^j)^{\mathrm{T}}}}, \quad j = 1, 2, \ldots, m, \tag{10}$$

where k_i^j is the jth row of matrix K_i, and $\lambda_{\min}(Q)$ is the minimum eigenvalue of matrix Q. Notice that, by Assumption 3.2, $[\bar{\beta}_{\max}]_j > 1$, for all $j = 1, 2, \ldots, m$.

Theorem 3.1. *The system* (1) (*subject to Assumption* 3.1) *having PLC controller* (8) *computed under Assumption 3.2 with allowed over-saturation* $0 \leqslant \bar{\beta}_j \leqslant [\bar{\beta}_{\max}]_j$, $j = 1, 2, \ldots, m$, *is asymptotically stable for all* $x \in \mathscr{E}_1$ (*i.e. in the outermost ellipsoid considered*).

Proof. Given the hybrid nature of the control system (1), (8), in the sense that continuous states $(x(t))$ and discrete states (related to the switching strategy) are present, we choose a piecewise quadratic candidate Lyapunov function of the form:

$$V(x) = x^{\mathrm{T}} P_i x \quad \text{for } x \in \mathscr{C}_i, \quad i = 1, 2, \ldots, N. \tag{11}$$

From (1) and (8), we see that the time derivative of the Lyapunov function inside cell $\mathscr{C}_i$ is

$$\begin{aligned} \dot{V}(x) &= [Ax + B\,\mathrm{sat}_{\tilde{\Delta}}(-K_i x)]^{\mathrm{T}} P_i x \\ &\quad + x^{\mathrm{T}} P_i [Ax + B\,\mathrm{sat}_{\tilde{\Delta}}(-K_i x)] \\ &= x^{\mathrm{T}}(A^{\mathrm{T}} P_i + P_i A)x + [\mathrm{sat}_{\tilde{\Delta}}(-K_i x)]^{\mathrm{T}} B^{\mathrm{T}} P_i x \\ &\quad + x^{\mathrm{T}} P_i B\,[\mathrm{sat}_{\tilde{\Delta}}(-K_i x)] \\ &= -x^{\mathrm{T}} Q x + \sum_{j=1}^m r_i^j |u_j| [|u_j| - 2\,\mathrm{sat}_{\Delta_j}(|u_j|)] \end{aligned} \tag{12}$$

for $x \in \mathscr{C}_i$, $i = 1, 2, \ldots, N$, where the last equality follows from (3) and the definitions of the $\mathrm{sat}_{\tilde{\Delta}}(\cdot)$ function in (2) and the diagonal matrix $R_i = \mathrm{diag}(r_i^1, r_i^2, \ldots, r_i^m)$. Here, we have denoted, by u_j, the jth element of $u = -K_i x$, i.e.

$$u_j = -\frac{1}{r_i^j} b_j^{\mathrm{T}} P_i x = -k_i^j x. \tag{13}$$

By the construction of $\mathscr{E}_i$, it can be readily seen from (7) that

$$|u_j| \leqslant (1 + \bar{\beta}_j)\Delta_j \leqslant (1 + [\bar{\beta}_{\max}]_j)\Delta_j, \quad \forall x \in \mathscr{C}_i \subset \mathscr{E}_i \tag{14}$$

for $j = 1, 2, \ldots, m$, where $[\bar{\beta}_{\max}]_j > 1$ is given by (10).

We will next consider the general case where $0 \neq x \in \mathscr{C}_i$ is such that $|u_{j_s}| \leqslant 2\Delta_{j_s}$ for j_s in a subsequence $j_s \in \{j_1, j_2, \ldots, j_p\}$, and $2\Delta_{j_s} < |u_{j_s}| \leqslant (1 + [\bar{\beta}_{\max}]_{j_s})\Delta_{j_s}$ for j_s in the complementary subsequence $j_s \in \{j_{p+1}, j_{p+2}, \ldots, j_m\}$, where $0 \leqslant p \leqslant m$, and $\{j_1, j_2, \ldots, j_p\} \cup \{j_{p+1}, j_{p+2}, \ldots, j_m\} = \{1, 2, \ldots, m\}$, $\{j_1, j_2, \ldots, j_p\} \cap \{j_{p+1}, j_{p+2}, \ldots, j_m\} = \emptyset$. Then, by the Cauchy–Schwarz inequality we obtain

$$|u_{j_s}|^2 = |k_i^{j_s} x|^2 > 4\Delta_{j_s}^2, \quad s = p+1, p+2, \ldots, m$$

$$\Rightarrow -x^{\mathrm{T}}x < \frac{-4\Delta_{j_s}^2}{k_i^{j_s}(k_i^{j_s})^{\mathrm{T}}}, \quad s = p+1, p+2, \ldots, m$$

$$\Rightarrow -x^{\mathrm{T}}x < -\sum_{s=p+1}^{m} \frac{r_i^{j_s}}{(\sum_{l=p+1}^{m} r_i^{j_l})} \frac{4\Delta_{j_s}^2}{k_i^{j_s}(k_i^{j_s})^{\mathrm{T}}}, \tag{15}$$

and $\mathrm{sat}_{\Delta_{j_s}}(|u_{j_s}|) = \Delta_{j_s}$, $s = p+1, p+2 \ldots, m$. Therefore, $\dot{V}(x)$ in (12) can be majorized as

$$\dot{V}(x) < \sum_{s=1}^{p} r_i^{j_s} |u_{j_s}| [|u_{j_s}| - 2\,\mathrm{sat}_{\Delta_{j_s}}(|u_{j_s}|)] + \sum_{s=p+1}^{m} r_i^{j_s} \left[|u_{j_s}|^2 - 2|u_{j_s}|\Delta_{j_s} - \frac{4\lambda_{\min}(Q)\Delta_{j_s}^2}{(\sum_{l=p+1}^{m} r_i^{j_l}) k_i^{j_s}(k_i^{j_s})^{\mathrm{T}}} \right] \tag{16}$$

for $0 \neq x \in \mathscr{C}_i$.

Since $|u_{j_s}| \leqslant 2\Delta_{j_s}$ for $s = 1, 2, \ldots, p$, we obtain that the terms between square brackets in the first summation in (16) satisfy

$$|u_{j_s}| - 2\,\mathrm{sat}_{\Delta_{j_s}}(|u_{j_s}|) \leqslant 0 \quad \text{for } s = 1, 2, \ldots, p. \tag{17}$$

It is easy to see that the quadratic terms between square brackets in the second summation in (16) are non-positive if and only if

$$|u_{j_s}| \leqslant \left(1 + \sqrt{1 + \frac{4\lambda_{\min}(Q)}{(\sum_{l=p+1}^{m} r_i^{j_l}) k_i^{j_s}(k_i^{j_s})^{\mathrm{T}}}} \right) \Delta_{j_s}, \tag{18}$$

$s = p+1, p+2, \ldots, m$. This is, in turn, true since, from the construction of $\mathscr{E}_i$ we have $|u_{j_s}| \leqslant (1 + [\bar{\beta}_{\max}]_{j_s})\Delta_{j_s}$ for all $j_s \in \{1, 2, \ldots, m\}$, and

$$[\bar{\beta}_{\max}]_{j_s} \leqslant \sqrt{1 + \frac{4\lambda_{\min}(Q)}{(\sum_{l=p+1}^{m} r_i^{j_l}) k_i^{j_s}(k_i^{j_s})^{\mathrm{T}}}}, \quad 0 \leqslant p \leqslant m. \tag{19}$$

Therefore we see that, under the assumptions of the theorem, we have

$$\dot{V}(x) < 0 \quad \text{for } 0 \neq x \in \mathscr{C}_i, \ \ i = 1, \ldots, N. \tag{20}$$

We conclude that the trajectories in each cell $\mathscr{C}_i$ approach the origin with a monotonic decrease in V along the trajectory. Since the ellipsoids are nested and all contain the origin, this means that the trajectories will cross the cell boundaries as they approach the origin. Finally, the trajectories will enter the smallest ellipsoid corresponding to ρ_N, where asymptotic convergence to the origin is assured by (20). □

Corollary 3.1. *The ellipsoids* $\mathscr{E}_i \triangleq \{x : x^{\mathrm{T}} P_i x \leqslant \rho_i\}$ *are positively invariant sets for system* (1) *under the control* $u(t) = -K_i x(t)$, *i.e., for any initial condition* $x_0 = x(t_0)$ *such that* $x_0^{\mathrm{T}} P_i x_0 \leqslant \rho_i$, *then for all* $t \geqslant t_0$, $x(t)^{\mathrm{T}} P_i x(t) \leqslant \rho_i$, *where* $x(t)$ *is the solution of* (1), *with control* $u(t) = -K_i x(t)$.

Proof. To establish this fact, note that for any $x(t)$ along a trajectory that satisfies $x(t)^{\mathrm{T}} P_i x(t) \leqslant \rho_i$, inequality (20) is satisfied. This means that the trajectories will never leave the ellipsoid $\mathscr{E}_i \triangleq \{x : x^{\mathrm{T}} P_i x \leqslant \rho_i\}$. □

Remark 3.1. Note that the inequality (20) also eliminates the possibility of chattering when switching at the boundaries $V(x) = x^{\mathrm{T}} P_i x = \rho_i$, since all the trajectories will head away from the boundaries as they approach the origin (Wredenhagen & Bélanger, 1994).

Notice that it is not possible to know $[\bar{\beta}_{\max}]_j$, $j = 1, \ldots, m$, *beforehand* since the design starts with the allowed over-saturation indices $\bar{\beta}_j$ and with them, and the ellipsoids radii ρ_i, we determine the diagonal matrices R_i as the largest values such that (7) is fulfilled. The R_i matrices, in turn, determine $[\bar{\beta}_{\max}]_j$ in (10). Therefore, the stability conditions of Theorem 3.1, i.e. $\bar{\beta}_j \leqslant [\bar{\beta}_{\max}]_j$, must be checked *after* the design has been carried out. In all cases, stability can be guaranteed, under the assumptions of Theorem 3.1 (Theorem 3.2 below), by choosing $\bar{\beta}_j \leqslant 1$ ($\bar{\beta}_j < 1$), $j = 1, \ldots, m$. In the cases where it is desirable to maximise the degree of allowed over-saturation, it could be required to perform some experimentation in the selection of $\bar{\beta}_j$. This is not problematic since the calculations are done off-line during the design. Also, in the scalar-input case, the design procedure can be slightly modified (in a similar form as done in Example 5.1 below) in such a way that $[\bar{\beta}_{\max}]$ can be computed *before* the selection of the over-saturation index.

We will now replace Assumption 3.2 by the following:

Assumption 3.3. The design matrix Q in (3) is non-negative definite, denoted as $Q \geqslant 0$. We also make the standard assumption in linear quadratic optimal control that the pair (A, D) is completely observable, where D is any matrix such that $DD^{\mathrm{T}} = Q$ (see, e.g., Anderson & Moore, 1989).

Notice that, from this assumption, we have in (10) that $[\bar{\beta}_{\max}]_j = 1$, for all $j = 1, 2, \ldots, m$. We then have the following theorem.

Theorem 3.2. *The system* (1), *subject to Assumption* 3.1, *with PLC controller* (8) *computed under Assumption* 3.3 *and having allowed over-saturation* $0 \leqslant \bar{\beta}_j < 1$, $j = 1, 2, \ldots, m$, *is asymptotically stable for all* $x \in \mathscr{E}_1$.

Proof. The procedure is similar to that used in the proof of Theorem 3.1. We choose a Lyapunov function as in (11) and get the time derivative $\dot{V}(x)$ as in (12). By the construction of $\mathscr{E}_i$, we have from (7) that

$$|u_j| \leqslant (1+\bar{\beta}_j)\Delta_j < 2\Delta_j, \quad \forall x \in \mathscr{C}_i \subset \mathscr{E}_i \tag{21}$$

for $j = 1,2,\ldots,m$. Hence, in (12) we obtain

$$|u_j| - 2\,\mathrm{sat}_{\Delta_j}(|u_j|) < 0 \quad \text{for } 0 \neq x \in \mathscr{C}_i \tag{22}$$

for $j = 1,2,\ldots,m$.

Since Q is non-negative definite, certainly $\dot{V}$ is non-positive, but to conclude asymptotic stability we must show that $\dot{V}$ is not identically zero along trajectories other than the trivial trajectory, $x(t) = 0$. Let us examine this condition. In order to have $\dot{V}(x) = 0$ in (12) we must have:

$$x(t)^{\mathrm{T}} Q x(t) = 0 \tag{23}$$

and

$$|u_j|(|u_j| - 2\,\mathrm{sat}_{\Delta_j}(|u_j|)) = 0, \tag{24}$$

$j = 1,2,\ldots,m$. Condition (23) is equivalent to

$$D^{\mathrm{T}} x(t) = 0, \tag{25}$$

where D is any matrix such that $DD^{\mathrm{T}} = Q$.

From (22), we have that condition (24) is equivalent to $|u_j| = 0$, for $j = 1,2,\ldots,m$, which means that the trajectory, $x(t)$, is the free response of the system. Hence, by taking successive derivatives of (25) we get $[D\ A^{\mathrm{T}}D\ \cdots\ (A^{\mathrm{T}})^{n-1}D]^{\mathrm{T}} x(t) = 0$, and by Assumption 3.3, we have that $x(t)$ must be $x(t) = 0$. Therefore, $\dot{V}(x) < 0$ for $0 \neq x \in \mathscr{C}_i$, $i = 1,\ldots,N$, and we conclude, as in Theorem 3.1, that the system is asymptotically stable. □

Remark 3.2. Note that, the above proof covers system (1) having the standard PLC controller of Wredenhagen and Bélanger (1994). This is simply a particular case of Theorem 3.2, when $\bar{\beta}_j = 0$ for $j = 1,2,\ldots,m$.

Example 3.1. In this example, we compare the proposed switching controller (8) having allowed over-saturation $\bar{\beta} > 0$ with the standard PLC controller presented in Wredenhagen and Bélanger (1994). For this purpose, we consider the system given in Example 1 of Wredenhagen and Bélanger (1994), i.e. a simple pendulum having state space description

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{sat}_{\Delta}(u) \tag{26}$$

with initial condition $(\theta, \dot{\theta}) = (54^\circ, 20^\circ\ \mathrm{s}^{-1})$ and saturation bound $\Delta = 5$. In Wredenhagen and Bélanger (1994, Example 1), a PLC controller consisting of 6 gains computed for $Q = I_{2\times 2}$ and $\rho_1 = 7.07$, $\rho_i = \rho_1(\Delta\rho)^{(i-1)}$, $i = 2,\ldots,6$, with a radius reduction factor of $\Delta\rho = 1/2$, is used. For comparison purposes, we have used the same gains as in Wredenhagen and Bélanger (1994, Example 1), and considered a controller having allowed over-saturation equal to the maximum over-saturation index allowed by Theorem 3.1, i.e., $\bar{\beta} = [\bar{\beta}_{\max}]$, where $[\bar{\beta}_{\max}] = 2.1191$ is computed from (10). In Fig. 1 we show the results obtained with the PLC of Wredenhagen and Bélanger (1994, Example 1), and the results obtained with a PLC with the maximum allowed over-saturation $\bar{\beta} = [\bar{\beta}_{\max}] = 2.1191$ (PLC/OS). In Fig. 1 (a) and (b) it can be seen that the PLC/OS has a faster response compared with the standard PLC controller. Fig. 1(c) shows the controls of both schemes. Note that the PLC/OS stays saturated at -5 during approximately 0.5 s, whereas the PLC avoids saturation via switching. In Wredenhagen and Bélanger (1994), the performance measure used for comparison between the PLC controller and fixed gain controllers is defined as the state energy cost

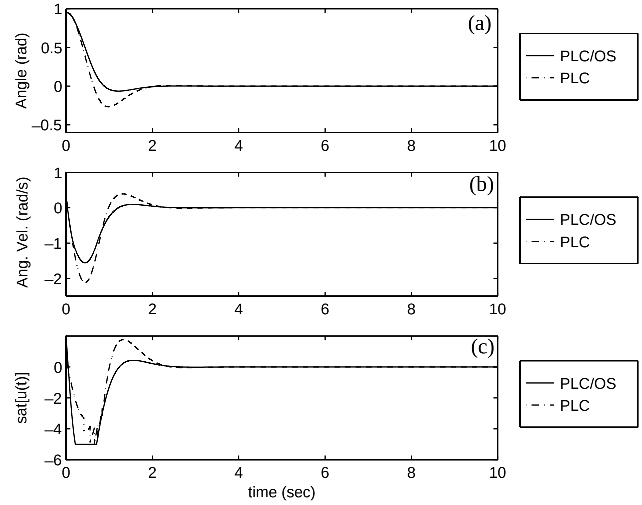


Fig. 1. Comparison of PLC control and PLC with allowed over-saturation (PLC/OS).

$$J_N(\Delta\rho) = \int_{t_0}^{+\infty} x^{\mathrm{T}}(t) Q x(t)\,\mathrm{d}t, \tag{27}$$

which is a function of the radius reduction factor $\Delta\rho$ of the ellipsoids and of the number of switching regions N. The cost obtained in Wredenhagen and Bélanger (1994) with the fixed initial gain $K_1 = [0.1118\ 1.5724]$ is $J_1 = 3.2664$, whereas with a PLC of 6 switched gains the cost is $J_6(0.5) = 2.3907$. With the maximum allowed over-saturation $\bar{\beta} = [\bar{\beta}_{\max}]$, the cost is $J_6(0.5) = 1.5233$. For comparison, we have also computed the cost with allowed over-saturation of $\bar{\beta} = 0.4$. In this case the cost is $J_6(0.5) = 1.9118$. In Fig. 2, we show the state trajectories in the phase plane for the fixed gain control ($K_1 = [0.1118\ 1.5724]$), for the standard PLC control, for the PLC/OS control with $\bar{\beta} = 0.4$, and for the PLC/OS control with $\bar{\beta} = [\bar{\beta}_{\max}] = 2.1191$. Also shown are the switching cells $\mathscr{C}_i$ for the PLC controller. (In the case of the controllers with allowed over-saturation the switching cells are bigger than the ones shown.) We have

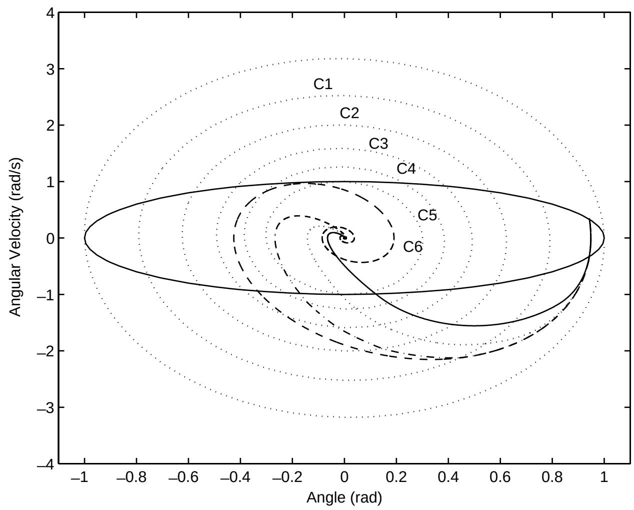


Fig. 2. State trajectories in phase plane for the simple pendulum. Solid: $\bar{\beta}=2.1191$. Dotted: $\bar{\beta}=0.4$. Dash-dot: $\bar{\beta}=0$ (standard PLC). Dashed: fixed gain K_1. Also shown: switching cells (dotted) and unit circle (solid).

also included the unit circle (surface of all initial conditions in the example considered in Wredenhagen & Bélanger, 1994).

4. PLC combined with low- and high-gain feedback control

In Lin et al. (1997), the authors have proposed an alternative modification of the PLC scheme aimed at obtaining a better utilisation of the available control authority. Lin et al. (1997) present a combination of the PLC controller and the low- and high-gain control (LHG) (see, e.g., Lin, 1998), aimed at rejecting input-additive disturbances and making the design robust to input-additive uncertainties in the context of actuator rate-saturation. The model, with actuator rate-saturation, is transformed via pre-feedback, into a model with position-saturation and the PLC and LHG techniques are then applied. In order to recast this scheme within the framework of Sections 2 and 3, we will present the simpler case, namely nominal model subject to position-saturation.

Given a system as in (1), the design starts with the PLC controller presented in Section 2. Then, to each of the control laws (8) a 'high gain' component is incorporated by multiplying the gains with a scaling factor $(1+k)$ with $k \geqslant 0$. Thus, the combined PLC/LHG control law is

$$\hat{u} = -(1+k)K_i x \quad \text{for } x \in \mathscr{C}_i, \ i = 1, 2, \ldots, N, \tag{28}$$

where $k \geqslant 0$ is a design parameter.

In Theorem 3.1 of Lin et al. (1997) it is proved that there exists a $k^* \geqslant 0$ such that the system with control (28) is asymptotically stable and is ultimately bounded in the presence of disturbances for all $k \geqslant k^*$. In particular, stability can be proven in the absence of disturbances, $\forall k \geqslant 0$ by using a piecewise quadratic Lyapunov function as has been done in Theorem 3.2 above. In Lin et al. (1997) the case $Q = I_{n\times n}$ is considered, in the next lemma we assume, more generally, that $Q \geqslant 0$. The proof of the lemma follows closely that of Theorem 3.2 and the result in Lin et al. (1997).

Lemma 4.1. *The system* (1) (*subject to Assumption* 3.1) *having PLC/LHG controller* (28) *computed under Assumption* 3.1 *with* $k \geqslant 0$, *is asymptotically stable for all* $x \in \mathscr{E}_1$.

It is argued in Lin et al. (1997) that the PLC/LHG controller inherits the advantages of both, the standard PLC and LHG techniques, while avoiding their disadvantages. In particular, within the PLC framework, increasing the feedback gain in a piecewise fashion as the trajectories converge towards the origin, results in fast transient speed for all states. Moreover, the LHG design, by providing a high-gain component, provides good utilisation of the available actuator authority and speeds up the transient response. It is shown in Lin et al. (1997) that, both the degree of disturbance rejection and the tolerance to actuator inaccuracies, are increased by increasing the value of the design parameter k. In practise however, k cannot be increased arbitrarily without invoking undesirable characteristics. For example, the LHG design, in common with all high-gain feedback laws, has an inherent sensitivity to measurement noise (see, e.g., Lin, 1998). Furthermore, as k increases, the saturation function $\mathrm{sat}_\Delta((1+k)K_i x)$ approximates a relay function $\Delta \times \mathrm{sgn}(K_i x)$ (see, e.g., Johansson, Rantzer, & Åström, 1999). Hence, various phenomena associated with relay feedback systems are likely to become important, e.g., fast switching behaviour, limit cycles, and chattering modes in the vicinity of the zero error region (see e.g., Johansson et al., 1999; Ledwich, 1995).

5. Combining PLC with over-saturation and scaling

Sections 3 and 4 have discussed two ideas for improving the performance of the PLC scheme. These schemes modify the basic PLC structure, (i) by allowing over-saturation in the switching algorithm, by which we mean extending each linear control up to regions beyond the constraints levels, and; (ii) by adding, to each linear control, a high-gain component, achieved via a scaling operation. In both cases it has been shown that the system retains asymptotic stability. The natural question is then, which of the above methods is more beneficial. Up to the present, there is no definite answer to this question, since many factors are involved. On the one hand, the performance of the standard PLC (and, as a consequence, those of (i) and (ii)) is not fully understood, i.e. the choice of design parameters ($\rho_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, and $Q \in \mathbb{R}^{n\times n}$) in order to optimise the performance is an open problem (see, Wredenhagen & Bélanger, 1994). On the other hand, the comparison of methods (i) and (ii) depends on initial conditions, that is, for some initial conditions one method may give better performance and vice versa. However, simulation results presented below show that a synthesis of both methods yields performance improvements in virtually all cases. This is illustrated in Example 5.1 below.

In this section we explore the combination of the PLC controller with both, allowed over-saturation and a high-gain scaling component. We will first show that the combined controller guarantees asymptotic stability. This is a straightforward consequence of the results in Sections 3 and 4.

Lemma 5.1. *The system* (1) (*subject to Assumption* 3.1) *with PLC/LHG controller* (28) *computed under Assumption* 3.2 (*Assumption* 3.3) *with* $k \geqslant 0$ *and with allowed over-saturation* $0 \leqslant \bar{\beta}_j \leqslant [\bar{\beta}_{\max}]_j$ $(0 \leqslant \bar{\beta}_j < 1)$, $j = 1, 2, \ldots, m$, *is asymptotically stable with* $\mathscr{E}_1$ *contained in its domain of attraction.* ($[\bar{\beta}_{\max}]_j > 1$ *is defined in* (10).)

Proof. Choose the Lyapunov function (11) whose time derivative $\dot{V}(x)$ for $x \in \mathscr{C}_i \subset \mathscr{E}_i$, $i = 1, 2, \ldots, N$, is given by

$$\dot{V}(x) = -x^{\mathrm{T}} Q x + \sum_{j=1}^{m} r_i^j |u_j| [|u_j| - 2\,\mathrm{sat}_{\Delta_j}((1+k)|u_j|)], \tag{29}$$

where u_j are given by (13).

Let us consider first the case of Assumption 3.2. Assume $|u_{j_s}| \leqslant 2\Delta_{j_s}$ for j_s in a subsequence $j_s \in \{j_1, j_2, \ldots, j_p\} \subset \{1, 2, \ldots, m\}$, and $2\Delta_{j_s} < |u_{j_s}| \leqslant (1 + [\bar{\beta}_{\max}]_{j_s})\Delta_{j_s}$ for j_s in the complementary subsequence $j_s \in \{j_{p+1}, j_{p+2}, \ldots, j_m\} \subset \{1, 2, \ldots, m\}$. A similar analysis to that used in the proof of Theorem 3.1 gives that $\dot{V}(x)$ can be majorised as:

$$\begin{aligned} \dot{V}(x) < {} & \sum_{s=1}^{p} r_i^{j_s} |u_{j_s}| [|u_{j_s}| - 2\,\mathrm{sat}_{\Delta_{j_s}}((1+k)|u_{j_s}|)] \\ & + \sum_{s=p+1}^{m} r_i^{j_s} \left[|u_{j_s}|^2 - 2|u_{j_s}|\Delta_{j_s} \right. \\ & \left. - \frac{4\lambda_{\min}(Q)\Delta_{j_s}^2}{(\sum_{l=p+1}^{m} r_i^{j_l}) k_i^{j_s} (k_i^{j_s})^{\mathrm{T}}} \right] \end{aligned} \tag{30}$$

for $0 \neq x \in \mathscr{C}_i$. Since $|u_{j_s}| \leqslant 2\Delta_{j_s}$ for $s = 1, 2, \ldots, p$, we obtain that $[|u_{j_s}| - 2\,\mathrm{sat}_{\Delta_{j_s}}((1+k)|u_{j_s}|)] \leqslant 0$, $\forall k \geqslant 0$, for each $s = 1, 2, \ldots, p$. Also, from (19) and the fact that $|u_{j_s}| \leqslant (1 + [\bar{\beta}_{\max}]_{j_s})\Delta_{j_s}$ for all $j_s \in \{1, 2, \ldots, m\}$, we have that the quadratic terms between square brackets in the second summation are non-positive. We conclude, as in Theorem 3.1, that the system is asymptotically stable. Finally, the case of Assumption 3.3 follows essentially as in the proof of Theorem 3.2. □

Example 5.1. In this example we explore to what extent the performance can be improved by combining the PLC controller with over-saturation and with a high-gain scaling component. Consider the system $G(s) = 1/s\,(s + 0.01)$ subject to input saturation with saturation bound $\Delta = 1$, and with the following state-space realisation:

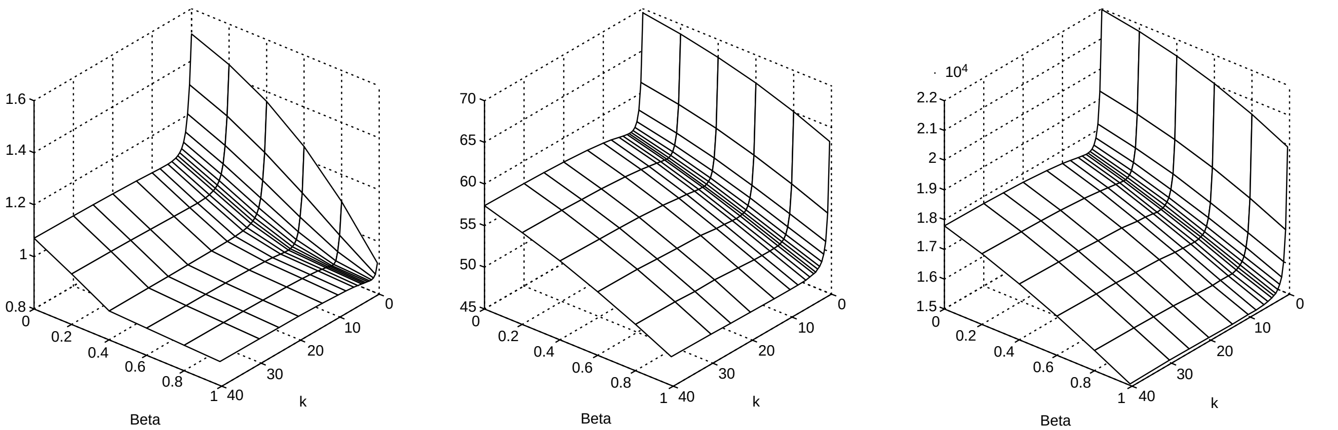
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -0.01 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{sat}_{\Delta}(u); \quad y = x_1. \tag{31}$$

The output is desired to track a constant reference r with a control $u = -\mathrm{sat}_{\Delta}(K\tilde{x})$, where the state has been modified by using a standard technique in order to transform the problem into a regulation problem (i.e., $\tilde{x}_1 = x_1 - r$, $\tilde{x}_2 = x_2 - 0.01\,r$). We use the switching controller (28), which consists of six gains $K_1, \ldots, K_6$ computed using (3) and (4) with $Q = [1\ 0; 0\ 0]$ and the following input weights: $R_1 = 5000$, $R_2 = 500$, $R_3 = 50$, $R_4 = 5$, $R_5 = 0.5$, and $R_6 = 0.05$. Notice that the design carried out in this example constitutes a slight modification of the design explained in Section 3. This particular design, which starts by choosing $R_1, \ldots, R_6$ (instead of $\rho_1, \ldots, \rho_6$), simplifies the computations since no iteration is needed in order to satisfy (7). (However, this variation of the design can only be performed for the scalar-input case.) In this case, the ellipsoids radii such that (7) is satisfied are computed, for a given R_i, from

$$\rho_i = \frac{(1 + \bar{\beta})^2 \Delta^2 R_i^2}{B^{\mathrm{T}} P_i B}, \quad i = 1, 2, \ldots, N. \tag{32}$$

In order to guarantee asymptotic stability, we have used an over-saturation index of up to $\bar{\beta} = 0.99$ to satisfy the conditions of Lemma 5.1. The asymptotic stability of the transformed system guarantees reference tracking for the original system. For comparison purposes we consider the state energy cost as in Eq. (27) of Example 3.1, i.e. we consider only the fix component of the design cost (5) (see, Wredenhagen and Bélanger, 1994). For this particular problem, and making a transformation into a zero tracking problem, the cost is given by $J_6(k, \bar{\beta}) = \int_{t_0}^{+\infty} [x_1(t) - r]^2\, \mathrm{d}t$. The cost $J_6(k, \bar{\beta})$ obtained for different values of k and $\bar{\beta}$ is shown in Fig. 3(a)–(c) for three reference values, $r = 1$, 5 and 50, respectively. In the simulations, the scaling factor k in (28) was allowed to change from 0 to 40. Beyond these values, no further improvement in the cost was obtained or the cost started to deteriorate. The over-saturation index $\bar{\beta}$ was changed between 0 and 0.99. In Fig. 3 it can be noticed that some combination of both, scaling and over-saturation, gives the lowest cost for the three operating conditions considered (compare with $J_6(0, 0)$, i.e. the cost for the standard PLC). Notice from the figure that a choice of $k = 5$ and $\bar{\beta} = 0.99$ gives a good compromise for the three values of r.

In the sequel, we give a heuristic explanation of the effects of allowing over-saturation and scaling on the performance



(a) r = 1 (b) r = 5 (c) r = 50

Fig. 3. Cost $J_6(k, \bar{\beta})$ for different values of $k \in [0, 40]$ and $\bar{\beta} \in [0, 1)$.

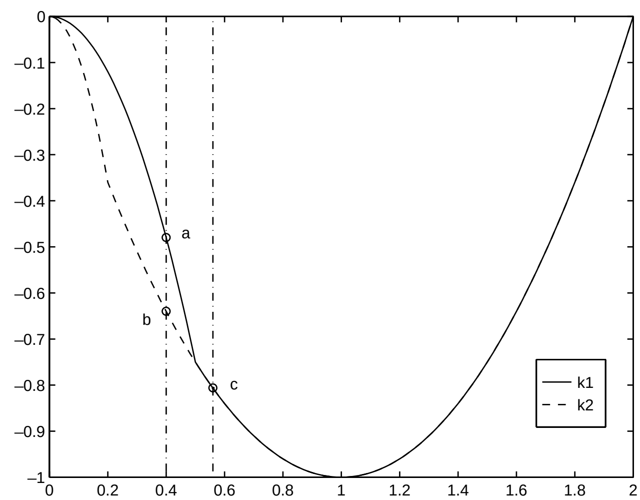


Fig. 4. Plot of $|\tilde{u}_j|^2 - 2|\tilde{u}_j|\mathrm{sat}_1((1+k_i)|\tilde{u}_j|)$ vs. $|\tilde{u}_j|$, for k_1, k_2 ($k_1 < k_2$).

of the PLC controller. For this purpose we use the derivative of the Lyapunov function as an indicator of the transient speed of the states (a similar approach has been used, in the case of the LHG controller, in Lin et al., 1997). In Fig. 4 we show a *normalised plot* of the terms appearing in the sum of the Lyapunov derivative (29), i.e., $|\tilde{u}_j|^2 - 2|\tilde{u}_j|\mathrm{sat}_1((1 + k_i)\,|\tilde{u}_j|)$, as a function of $|\tilde{u}_j|$ (with $\tilde{u}_j \triangleq u_j/\Delta_j$). The effect of scaling the control input can be appreciated by comparing the curves for k_1 and k_2 ($k_2 > k_1$) in Fig. 4; i.e. the transient speed for states not in the null-space of $B^{\mathrm{T}}P_i$ will increase as the value of k increases (see, Lin et al., 1997). Also, the effect of allowing over-saturation can be appreciated from Fig. 4. In the PLC/LHG controller, the switching always occurs when $|\tilde{u}_j| \leqslant 1$ (i.e. $|u_j| \leqslant \Delta_j$). Notice that the switching at $|\tilde{u}_j| = 1$ can only occur at two possible points on the switching ellipsoid and, thus, for most of the trajectories, the switching takes place at points where $|\tilde{u}_j| < 1$; for example, at points of the state space corresponding to points ‘a’ and ‘b’ in Fig. 4, for k_1 and k_2, respectively. However, it can be seen in the figure that it is more beneficial to switch at a higher value of $|\tilde{u}_j|$, for instance at point ‘c’. The latter is the case when the controller is allowed to have over-saturation, which in turn results in bigger ellipsoids leading to larger values of $|\tilde{u}_j|$ for the same value of x. In fact, even in those cases where the switching occurs at, or near, $|\tilde{u}_j| = 1$ in the PLC/LHG controller, it is beneficial to allow some level of over-saturation, since the function $\dot{V}(x)$ will then take more negative values during longer periods of time.

6. Robust design

In Section 5 we have shown that two existing ideas used to improve the performance of the basic PLC design; namely, over-saturation and scaling, can be combined into a more general strategy. We have argued that this combination, when properly chosen, yields performance improvements as compared to each modification being carried separately. Example 5.1 illustrates this assertion. We have also shown that the resulting combined controller guarantees asymptotic stability of the closed-loop system. One may ask how robust is the proposed scheme to plant uncertainties and how effective in achieving disturbance rejection. Robust stability in the presence of input-additive uncertainties has been established in Lin et al. (1997) for the PLC/LHG design (described in Section 4 above). A similar analysis can also be performed for the controller presented in Section 5, namely, the PLC/LHG controller combined with allowed over-saturation. For plants with uncertainties in the A-matrix we have shown in De Doná et al. (1999) that, at least in the single input case, the design with allowed over-saturation (Section 3) can be modified to guarantee robust stability, for an allowed over-saturation of up to 100% ($\bar{\beta} \leqslant 1$). One can anticipate that such a design could be extended to the combined controller of Section 5. In this section we will establish this extension for the case where $m = 1$, which is a consequence of the results presented in De Doná et al.

(1999) and in Section 5 above. We will also show that the allowed over-saturation can be extended beyond 100% with guaranteed robust stability.

Consider a class of uncertain systems described by

$$\dot{x}(t) = (A + DF(t)E)x(t) + B\,\mathrm{sat}_\Delta(u(t)), \tag{33}$$

where $x(t) \in \mathbb{R}^n$ is the state, $F(t) \in \mathbb{R}^{p\times q}$ is a matrix of uncertain parameters satisfying the bound $F^{\mathrm{T}}(t)F(t) \leqslant I_{q\times q}$, and $u(t) \in \mathbb{R}$ is the control input. We make the following assumption, (see Definition 2.1 in Petersen, 1987):

Assumption 6.1. System (33), without input saturation, is quadratically stabilisable.

The robust design starts with a sequence $\{R_i\}_{i=1}^N$ of N design parameters such that $R_1 > R_2 > \cdots > R_N > 0$, and an $n \times n$ design matrix $Q > 0$. For each R_i we consider the following Riccati equation (see, e.g., Petersen, 1987):

$$A^{\mathrm{T}}P_i + P_iA - P_iBR_i^{-1}B^{\mathrm{T}}P_i + \varepsilon_i P_iDD^{\mathrm{T}}P_i + \frac{1}{\varepsilon_i}E^{\mathrm{T}}E + Q = 0, \tag{34}$$

where ε_i is a positive constant. Then, for the smallest design parameter, R_N, we find a constant ε_N such that (34) has a positive-definite solution, denoted P_N. The existence of an ε_N such that the ARE has a positive-definite solution is guaranteed, independently of the choice of Q and R_N, by Assumption 6.1 (see, e.g., Theorem 3.3 in Petersen, 1987). For each of the $N-1$ remaining design parameters compute $\varepsilon_i = \varepsilon_N R_N/R_i$, and, with the pairs (R_i, ε_i) thus obtained, find the positive-definite solutions to the ARE (34), P_i, for $i = 1, 2, \ldots, N-1$ (the existence of these solutions has been established in De Doná et al., 1999). We then compute the sequence of gains $K_i = R_i^{-1}B^{\mathrm{T}}P_i$. The sequence of switching ellipsoids $\mathscr{E}_i \triangleq \{x : x^{\mathrm{T}}P_ix \leqslant \rho_i\}$ is designed such that inequality (7) is satisfied, i.e. the ellipsoids radii are computed from (32) with over-saturation index $\bar{\beta} > 0$, which is allowed to be up to $[\bar{\beta}_{\max}]$ (given by (10), with $m = 1$).

In De Doná et al. (1999) we have shown that the ellipsoids $\{\mathscr{E}_i\}_{i=1}^N$ are nested. This nesting property allows us to perform a partitioning of the state space region contained into the biggest ellipsoid in N cells: $\{\mathscr{C}_i\}_{i=1}^N$ defined as: $\mathscr{C}_i = \mathscr{E}_i \backslash \mathscr{E}_{i+1}$, for $i = 1, 2, \ldots, N-1$, and $\mathscr{C}_N = \mathscr{E}_N$. The controller is then defined by the switching strategy (28), where a high gain scaling component $(1 + k)$, $k \geqslant 0$, is included.

In the next lemma we prove the robust stability of this scheme, which we call robust PLC/LHG with allowed over-saturation.

Lemma 6.1. *The uncertain system* (33) (*subject to Assumption* 6.1) *and controller* (28) (*computed with the above robust PLC/LHG design*) *with* $k \geqslant 0$ *and with allowed over-saturation* $0 \leqslant \bar{\beta} \leqslant [\bar{\beta}_{\max}]$ *is asymptotically stable for all* $x \in \mathscr{E}_1$. ($[\bar{\beta}_{\max}]_j > 1$ *is defined by* (10), *with* $m = 1$.)

Proof. Choose a Lyapunov function (11) whose time derivative $\dot{V}(x)$ is given by

$$\begin{aligned}
\dot{V}(x) &= [(A + DFE)x + B\,\mathrm{sat}_\Delta(-(1+k)K_ix)]^{\mathrm{T}}P_ix \\
&\quad + x^{\mathrm{T}}P_i[(A + DFE)x + B\,\mathrm{sat}_\Delta(-(1+k)K_ix)] \\
&= x^{\mathrm{T}}[(A + DFE)^{\mathrm{T}}P_i + P_i(A + DFE)]x \\
&\quad - 2x^{\mathrm{T}}P_iB\,\mathrm{sat}_\Delta((1+k)K_ix)
\end{aligned} \tag{35}$$

for $x \in \mathscr{C}_i$, $i = 1, \ldots, N$. Then, from Claim 1 of Petersen (1987) we can find the following upper bound for the Lyapunov derivative:

$$\begin{aligned}
\dot{V}(x) &\leqslant x^{\mathrm{T}}\left(A^{\mathrm{T}}P_i + P_iA + \varepsilon_iP_iDD^{\mathrm{T}}P_i + \frac{1}{\varepsilon_i}E^{\mathrm{T}}E\right)x \\
&\quad - 2x^{\mathrm{T}}P_iB\,\mathrm{sat}_\Delta((1+k)K_ix) \\
&= -x^{\mathrm{T}}Qx + R_i|u|(|u| - 2\,\mathrm{sat}_\Delta((1+k)\,|u|))
\end{aligned} \tag{36}$$

for $x \in \mathscr{C}_i$, $i = 1, \ldots, N$, where the last equality follows from (34) and $u = -R_i^{-1}B^{\mathrm{T}}P_ix$. Notice that the last expression in (36) is the same as the right-hand side of Eq. (29) (with $m = 1$). Therefore, the same argument as in the proof of Lemma 5.1, for the case of Assumption 3.2, can be used to establish that the system is asymptotically stable. □

7. Conclusions

This paper has surveyed several ideas that can be used in the design of switching controllers to achieve high performance on linear systems having input saturation. The ideas evolve from an existing scheme known as the PLC controller. Two alternative ways of modifying the PLC controller have been described. These modifications include the concept of 'allowed over-saturation', and 'PLC/LHG design'. Both methods are aimed at forcing the control into saturation, thus making best use of the available control authority. A novel controller, which combines the above methods has also been presented. The combined controller has been shown to lead to performance advantages via a simulation example. Finally, a robust design has been presented. In all cases, the asymptotic stability of the resulting hybrid system has been established. There remain several interesting open research problems in this general area. For example, we have shown, via simulation, that performance is improved by the combination of over-saturation and scaling. It would be interesting to develop a mathematical framework for expressing this improvement. Also, it would be interesting to extend the robustness results to the multiple-input case and to other types of uncertainty.

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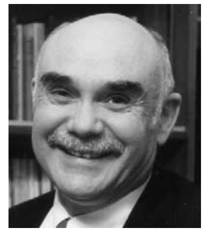

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