Brief paper

An analysis of signal transformation approach to triangular waveform tracking

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ABSTRACT

This paper is concerned with the mathematical analysis of the signal transformation approach to triangular waveform tracking. We provide necessary and sufficient conditions for stability and convergence of tracking error for a general class of plants and compensators. A procedure is offered to incorporate robustness against plant uncertainties and disturbances in the signal transformation method. Moreover, effectiveness of the method compared to an ordinary 2-DoF feedback control system is investigated. Simulation results are presented that reveal the conditions under which the incorporation of signal transformation blocks in a linear feedback loop may introduce control performance improvements.

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1. Introduction

Accurate tracking of a fast triangular waveform is one of the major challenges in scanning probe microscopy (Devasia, Eleftheriou, & Moheimani, 2007; Mahmood, Moheimani, & Liu, 2009) and other scanner-based devices such as optical scanners and selective laser sintering (SLS) machines (Yen, Yeh, Peng, & Lee, 2009; Zhiquang, Zude, Wu, & Youping, 2007). Raster scanning is also used in emerging probe-based data storage devices that require high-speed positioning with limited closed-loop bandwidth, due to measurement noise, actuator limitations, etc. (Pantazi et al., 2008, 2007). The triangular waveform contains all odd harmonics of the fundamental frequency. The high-frequency content in a fast triangular waveform can excite the resonant modes of the plant and subsequently degrade the tracking performance.

To track a fast triangular signal, high bandwidth closed-loop controllers have been implemented in many nanopositioning devices (Aphale, Devasia, & Moheimani, 2008; Bhikkaji, Ratnam, Fleming, & Moheimani, 2007; Bhikkaji, Ratnam, & Moheimani, 2007; Fleming, Wills, & Moheimani, 2008; Mahmood et al., 2009; Salapaka, Sebastian, Cleveland, & Salapaka, 2002; Schitter, Menold, Knapp, Allgöwer, & Stemmer, 2001; Yong, Aphale, & Moheimani, 2009). However, the scanning speed is limited when nanopositioners such as piezoelectric tubes are used to follow non-smooth triangular trajectories. The major causes of this limitation are hysteresis, thermal drift, sensor noise, uncertainty, and mechanical vibrations (Moheimani, 2008). Capacitive and inductive sensors are the commonly used sensors in nanopositioning systems due to their capability of providing a simple solution for non-contact, high-resolution measurement. These sensors typically have a noise density as low as 20 pm/√Hz (for a sensor with 100 µm range). For every hundredfold increment in the closed-loop system bandwidth, the position accuracy of a nanopositioning scanner will decrease by tenfold due to noise. This potentially degrades the resolution of the scanner, hindering it from performing positioning tasks that require subnanometer resolution. Hence, feedback control methods with limited closed-loop bandwidth are of considerable importance.

Command pre-shaping methods can be considered as a possible way for vibration suppression in an already designed closed-loop control system, leaving the closed-loop bandwidth of the measurement noise unaffected (Gilberta & Kolmanovskyb, 2002; Hyde & Seering, 1991; Kogiso & Hirata, 2009; Mimmi & Pennacchi, 2001; Pennacchi, 2004; Rhim, Lee, & Lim, 2005; Singer & Seering, 1990; Smith, 1958; Sugie & Suzuki, 2004; Suzuki & Sugie, 2008). However, these methods are not suitable for tracking of time-varying commands such as triangular waveforms and suffer from lack of robustness to plant uncertainties. Iterative learning control (ILC) can also be added as a feed-forward control action in a feedback system to improve the steady-state tracking error for repetitive references without altering the closed-loop bandwidth (Tien, Zou, & Devasia, 2005; Yen et al., 2005–1098/$ – see front matter Crown Copyright © 2011 Published by Elsevier Ltd. All rights reserved.

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2009). However, it may require a large number of iterations to converge or an excessive computational burden. Feedback control methods such as repetitive control (RC) for tracking of periodic references introduce large closed-loop bandwidths, which may not be acceptable in the presence of measurement noise. Moreover, the trade off between the tracking error and rejection of non-periodic disturbances in RC systems can cause problems (Aridogan, Shan, & Leang, 2009; Pipeleers, Demeulenaere, Al-Bender, De Schutter, & Swevers, 2009).

In Sebastian and Moheimani (2009), the concept of signal transformation was put forward as a novel method to track triangular waveforms in a nanopositioning system. The method showed significant closed-loop performance improvement compared with the ordinary one degree-of-freedom (1-DoF) feedback-control-system having a similar control bandwidth. However, only a second-order plant with no zeros and a double-integrator compensator were considered in Sebastian and Moheimani (2009), without any stability analysis.

In this paper, we provide an analysis of the signal transformation method, which leads to sufficient and necessary conditions for stability of the control system when tracking a triangular waveform. As in Sebastian and Moheimani (2009), the plant will be assumed to have a unity low-frequency gain. This condition is not so restricting as long as a stable feedback loop with an integrating compensator can be wrapped around the original plant. Furthermore, the analysis in this paper includes plants and compensators that have strictly proper transfer functions with arbitrary poles and zeros. The remainder of this paper is organized as follows. The signal transformation method is briefly reviewed in Section 2. Necessary and sufficient conditions for exponential stability of the system are formulated in Section 3. Steady-state behavior as well as sufficient and necessary conditions for convergence of the tracking error to bounded limits are quantified in Section 3.1. In Section 4, the signal transformation is applied after a robustifying PID control loop. Using simulations, the performance of signal transformation is evaluated compared to an ordinary 2-degrees-of-freedom (2-DoF) controller, having the same closed-loop noise bandwidth. Effects of parametric and unstructural uncertainties in the plant as well as input and output disturbances are also considered.

2. Signal transformation

Signal transformation method incorporates appropriate mappings between non-smooth signals (e.g. triangular waveforms) and smooth signals (e.g. ramps) in a control system to improve the tracking error while keeping the closed-loop bandwidth low to limit the projected measurement noise (Sebastian & Moheimani, 2009). The signal transformation method for control of a SISO plant is described by the hybrid control system shown in Fig. 1, where \( \Phi \) and \( \Phi^{-1} \) refer to the signal transformation mappings, which in the case of triangular signal tracking use piecewise constant gains \( g_1 \) and \( g_2 \), as well as biases \( b_1 \) and \( b_2 \), that can be presented in the following forms,

\[
\begin{align*}
g_1 &= g_2 = (-1)^{i-1} \\
b_1 &= (-1)^i b_2,
\end{align*}
\]

Here, \( a_0 \) is the amplitude of the desired triangular waveform \( x_d \), which has a period of \( 2T \), as shown in the left top insert in Fig. 1, and \( i \) is the index of half period defined as:

\[
i(t) = k, \quad \text{if } t \in [kT - T, kT), \quad k = 1, 2, 3, \ldots
\]

The signal transformation blocks, which use \( g_i \) and \( b_i \), can convert the non-smooth periodic triangular signal \( x_d \) to a smooth ramp signal denoted by \( r \) in the right top insert in Fig. 1. The signal transformation block between the plant and compensator does the reverse action, i.e. it can convert the smooth ramp signal into a non-smooth triangular signal. Under ideal circumstances, where the plant is a unity gain transfer function and its output is perfectly following the desired signal, the input/output signals at compensator block will be smooth signals with no breaks or discontinuities and the burden of providing appropriate non-smooth trajectories at the actuator, which demands a high control bandwidth in an ordinary feedback system, is on the signal transformation blocks. In this way, the compensator can be designed with a smaller closed-loop bandwidth in favor of rejecting the measurement noise without deteriorating the steady-state error.

3. Stability analysis

We assume that the plant and compensator dynamics are of degrees \( n_p \) and \( n_c \) and are described by linear-time-invariant state-space matrix sets \( [A_p, B_p, C_p] \) and \( [A_c, B_c, C_c] \) with \( X_p \) and \( X_c \) referring to the corresponding state vectors, respectively. The feedthrough matrices have been assumed zero. To start the analysis we merge the plant and its adjacent signal transformation blocks into a unified state-space model, called the equivalent plant. Hence, we wish to determine under what circumstances the simple control system shown in Fig. 2 is equivalent to the original hybrid control system in Fig. 1, i.e. with the same ramp signal \( r(t) \) in both control systems, the time histories of variables \( e \), \( v \), \( X_c \), and

![Fig. 1. Schematic diagram of the signal transformation method for triangular waveform tracking.](image-url)
In the equivalent plant, as described by (3) and Eq. (1)
\[ X_e := \frac{1}{g_1} (X_p + F), \quad F := A_p^{-1} B_p b_1, \]
provided that the gains and biases are constants (in the time interval)
satisfying the following relationships
\[ g_1 g_2 = 1, \quad b_2 - g_2 C_p A_p^{-1} B_p b_1 = 0, \]
and the equivalent state vector at the start of the time interval is initialized according to (3).

**Proof.** The signal transformations in Fig. 1 are described in the following forms.
\[ u = g_1 v + b_1; \quad y = g_2 x + b_2. \]

Consider the time interval \( t \in (iT - T, iT) \). Since the gain and bias signals are constant in this interval, the following state-space model is readily obtained using the plant state dynamics and Eq. (5), if the plant has no poles at the origin.
\[
\begin{align*}
\dot{X}_e &= A_p X_e + B_p y \\
y &= g_1 g_2 C_p X_e + (b_2 + g_2 \delta_0 b_1)
\end{align*}
\]
where \( \delta_0 = -C_p A_p^{-1} B_p \) is the DC gain of the plant. It is clear from Eq. (6) that we can replace the blocks between nodes \( v \) and \( y \) in Fig. 1 with the equivalent plant, as described by Fig. 2 and Eq. (3), and the control systems are equivalent if the conditions mentioned in Lemma 1 are satisfied. \( \square \)

Conditions (4) are satisfied with the selected gains and biases in Eq. (1) if the plant has a unity DC gain (\( \delta_0 = 1 \)). If the plant has a transfer function of the form:
\[ P_o(s) := \frac{x(s)}{u(s)} = \frac{\delta_0 + \delta_1 s + \cdots + \delta_{n-1} s^{n-1}}{1 + \epsilon_1 s + \cdots + \epsilon_{n-1} s^{n-1}}, \]
its state-space realization can be written in the following canonical form:
\[
A_p = \begin{bmatrix}
0_{(n_p-1)\times1}; & l_{n_p-1}; & -1 & 0 & \epsilon_1 & \epsilon_2 & \cdots & \epsilon_{n_p-1} & -\epsilon_n
\end{bmatrix},
\]
\[ B_p = \begin{bmatrix}
0_{(n_p-1)\times1}; & 1 & \epsilon_1 & \epsilon_2 & \cdots & \epsilon_{n_p-1}
\end{bmatrix},
\]
\[ C_p = \begin{bmatrix}
\delta_0 & \delta_1 & \cdots & \delta_{n_p-1}
\end{bmatrix}. \]

The overall state vector \( X \) of the equivalent closed loop system, defined as:
\[ X := \begin{bmatrix} X_e \\ X_c \end{bmatrix}, \]
obey the following state-space equation
\[
\begin{align*}
\dot{X} &= AX + Br \\
y &= CX
\end{align*}
\]
where
\[
A := \begin{bmatrix}
A_p & -B_p C_p A_p \\ -B_p C_p & A_c
\end{bmatrix},
B := \begin{bmatrix}
0_{n_p \times 1} \\ B_c
\end{bmatrix},
C := \begin{bmatrix}
C_p & 0_{1 \times n_c}
\end{bmatrix}. \]

The equivalent plant state \( X_e \) must be initialized by (3) at the start of each half period, which requires knowledge of plant state \( X_p \). To use the equivalent control system as a stand-alone machinery for analysis, appropriate formulas are necessary to update the equivalent state at the switching moments \( t = iT \). The following theorem gives the updating relationships at the switching moments.

**Theorem 2.** With the triangular reference signal shown in Fig. 1, signal transformation parameters (1), and unity DC gain for the plant, the overall state vector of the equivalent control system just before a switching moment obeys the recursive formula:
\[
X_e^{-} := X(T^{-}) = EX_0 + H,
\]
\[ X_e^{+} := X(t_{i+1})^{-} = \dot{X}^{-} + Ji + H, \quad i = 1, 2, 3, \ldots, \]
and the state just after a switching moment is updated using its value just before the switching moment as:
\[
X_e^{+} := X(t_{i+1}^{+}) = \hat{X}^{+} - Li, \quad i = 1, 2, 3, \ldots,
\]
where \( L \) is the constant \( (n_p + n_c) \times 1 \) vector:
\[
L := [2a_0, 0, \ldots, 0]^T,
\]
and
\[
\dot{i} := \begin{bmatrix}
-I_{n_p} & 0 \\ 0 & I_{n_c}
\end{bmatrix}, \quad \hat{X} := \begin{bmatrix}
X_p(0) \\ X_c(0)
\end{bmatrix},
\]
\[ E := e^{ijT}, \quad H := \begin{bmatrix}
1 - I & A_c^{-1} \cdot I \end{bmatrix} A^{-1} B a_0.
\]

**Proof.** In the original control system shown in Fig. 1, the gains and biases have discontinuous changes at the switching times \( t = iT \) (\( i = 1, 2, 3, \ldots \)), which makes the signals \( y \) and \( u \) discontinuous. However, the states and outputs of the plant and compensator \( X_e, X_p, v, x \) are continuous due to zero feedthrough matrices and the inherent integration actions in the compensator and plant state equations. Hence, the equivalent plant state \( X_e \) has discontinuities at the switching times because of \( g_1 \) and \( b_1 \) (see Eq. (3)). Thus, to maintain the equivalence of the simple control system shown in Fig. 2 with the original control system over time intervals longer than a half period, we have to intentionally incorporate appropriate jumps in the equivalent plant state \( X_e \) at each switching time, which can be described in the following form using Eq. (3).
\[
\Delta X_{i} := X_{i}^{+} - X_{i}^{-} = \left( \begin{bmatrix}
1 & -1 \\ g_1 & g_1
\end{bmatrix} \right) X_{i} + A_p^{-1} B_p \left( \begin{bmatrix}
b_1^{+} - b_1^{-} \\ g_1^{+} - g_1^{-}
\end{bmatrix} \right) \]
where the lowest subscript \( i \) for each variable refers to its value at the switching moment \( t = iT \) (\( i = 1, 2, 3, \ldots \)), and the plus and minus superscripts refer to the values of the corresponding variable at infinitesimal times just before and after the switching moment indicated by the lowest subscript, respectively, as \( X_{i} := X_p(iT) \), \( X_{i}^{+} := X_p((iT)^+) \), \( X_{i}^{-} := X_p((iT)^-) \), \( g_i^{+} := g_i(iT^+) \), \ldots. If we use Eq. (3) to replace \( X_p \) by \( g_1^{-} X_c^{-} - F^{-} \) in Eq. (19), the equivalent
plant state just after the switching moment can be described in terms of its value just before the switching moment as:

$$X_{i+1}^+ = \frac{1}{g_i} [g_i X_i^- + A_p^{-1} B_p (b_i^* - b_i^c)].$$  \hspace{1cm} (20)

The inverse of the plant state matrix is in the following form:

$$A_p^{-1} = \begin{bmatrix} -e_1 & -e_2 & -e_3 & \cdots & -e_{p-1} & -e_p \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & \cdots & \cdots & \cdots \\ \end{bmatrix}.$$  \hspace{1cm} (21)

Hence, $A_p^{-1} B_p = [-1, 0, \ldots, 0]^T$. Using (20) and the signal transformation gains and biases selected in (1) for the triangular reference, the equivalent plant state at the switching moments can be updated based on the following relationship:

$$X_i^+ = -X_i^- + L_i, \quad \text{for } i = 1, 2, 3, \ldots$$  \hspace{1cm} (22)

where $L$ is a $n_p \times 1$ vector defined similar to (15). Using (22) and the fact that the compensator state is continuous at the switching moments, the overall state $X_i$ just after the switching moment is easily obtained in the form of (14). Using (22), we can recursively obtain the analytical response of the equivalent control system shown in Fig. 2 to the transformed ramp input $r(t) = a_0 t / T$ if the plant has a unity DC gain, i.e., $\delta_0 = 0$. Using the commutativity of $A$ and $e^{At}$, the solution of the state $X$ from state-Eq. (10) in the time interval $t \in (T_i, T_{i+1})$ with ramp input $r = \frac{a_0 t}{T}$, in terms of the state just after the switching moment $t = T_i$, can be written in the following form.

$$X(t) = e^{At} X_i^+ + \left[ (e^{At} - I)(a_0 I + A^{-1} A) - a_0 t \right] A^{-1} B, \quad \text{for } t \in (T_i, T_{i+1}).$$  \hspace{1cm} (23)

where

$$t' := t - T_i, \quad \alpha := \frac{a_0}{T}, \quad X_i^+ := X_i(T_i).$$  \hspace{1cm} (24)

The initial state condition $X_0$ is composed of the plant and compensator initial states, with the selected with parameter values in (1). $X_0$ is equal to $X_p$ in the first half cycle. Using (23) with $i = 0$ and $t' = T$, the state just before the first switching moment is obtained in the form of (12). Using (23) with $t' = T$ and replacing $X_i^+$ by the right hand side of (14), the recursive Eq. (13) is readily obtained, which updates the state values from one switching moment to the next. \hspace{1cm} \Box

Eq. (13) defines a discrete-time LTI dynamic system with $\hat{A}$ as the state matrix, $[I, H]$ as the input matrix, and $[i, 1]^T$ as the input vector whose first vector element is a discrete-time ramp signal. Hence, a necessary condition for the closed-loop system to be free from exponentially unstable modes is that all eigenvalues of $\hat{A}$ are inside the unit disk. This condition is also a sufficient one because the state at the arbitrary time $t = T_i + t'$ depends on $X_i$ through Eqs. (14) and (23) and variable $t'$ is limited to $t' \in (0, T)$, which shows that if $X_i$ does not have any exponentially unstable mode, neither will $X(t)$. In the more general case, where the desired signal $x_d$ is an arbitrary bounded signal but the signal transformation parameters are kept as before with unity DC gain for the plant, Eq. (14) will not change but Eqs. (23), (12) and (13) can be represented in the following forms (for $t' \in (0, T)$):

$$X(t + T_i) = e^{At} X_i^+ + (e^{At} - I)A^{-1} B_0 \hat{d} + W(i, t'),$$  \hspace{1cm} (25)

$$X_i^+ = E_0 x_0 + W(0, T),$$  \hspace{1cm} (26)

$$X_{i+1}^+ = \hat{A} X_i^+ + j_i + W(i, T), \quad i = 1, 2, 3, \ldots, \quad (27)$$

where

$$W(i, t') = (e^{At'} - I)A^{-1} B_0 \frac{1}{2} (-1)^i + \int_0^{t'} e^{A(t'-\tau)} x_d(\tau + iT) d\tau B(-1)^i.$$  \hspace{1cm} (28)

Since vector $W(i, t')$ is bounded, because of the boundedness of $x_d$, the aforementioned condition about the absence of exponentially unstable modes is not restricted to the triangular waveform and is also valid for arbitrarily bounded reference inputs.

### Corollary 3

Assuming unity DC gain for the plant and signal transformation parameters (1), the hybrid control system is free from exponentially unstable modes if and only if the eigenvalues of matrix $\hat{A}$, defined in (18), are inside the unit circle.

Notice that the hybrid control system may have exponentially stable responses while the closed-loop state matrix $\tilde{A}$, defined in (11), may have unstable eigenvalues, which means under some circumstances, incorporation of signal transformation in an ordinary unstable feedback system may stabilize the closed-loop responses.

#### 3.1. Steady-state behavior with triangular reference

Contrary to ordinary control systems, in the hybrid system of Fig. 1, assumption of exponential stability does not imply BIBO stability (considering $x_d$ as the input and $x$ as the output). The signal transformation converts the original triangular reference into a ramp signal. However, it is not desirable for the plant states to grow linearly with time. The following theorem provides conditions under which the states of the plant remain bounded in steady-state conditions.

### Theorem 4

Assuming the triangular reference input in Fig. 1, unity DC gain for the plant, signal transformation parameters (1), and eigenvalues of matrix $\hat{A}$ inside the unit circle, the plant state in the hybrid control system will remain bounded if and only if either of the following conditions is satisfied.

$$\delta_c := -CA^{-1} B = 1$$  \hspace{1cm} (29)

$$P(t') := [I_{n_p}, 0] e^{At'} (I - \hat{A})^{-1} L + 0.5 L = 0, \quad \forall t' \in (0, T)$$  \hspace{1cm} (30)

**Proof.** Successive use of Eq. (13) leads to the following equation ($i = 1, 2, 3, \ldots$):

$$X_{i+1}^- = \hat{A} X_i^- + \sum_{k=0}^{i-1} \hat{A}^k [(i-k)I + H],$$  \hspace{1cm} (31)

which represents the state value just before a generic switching moment in terms of its value just before the first switching moment. We can simplify solution (31) if matrix $(I - \hat{A})$ is invertible. Given that the eigenvalues of $\hat{A}$ have magnitudes less than 1 this condition is met. Under such an assumption, the following equalities hold.

$$\sum_{k=0}^{i-1} (i-k) \hat{A}^k = \hat{A}^{i-1} - (I + 1)\hat{A} + i I (I - \hat{A})^{-2}$$  \hspace{1cm} (32)

$$\sum_{k=0}^{i-1} \hat{A}^k = (I - \hat{A})(I - \hat{A})^{-1}.$$  \hspace{1cm} (33)

Substituting the right-hand sides of Eqs. (32), (33) and (12) into Eq. (31) and replacing $i$ by $i - 1$, we obtain the closed-form formula.

$$X_i^- = [\hat{A}^i - i \hat{A} + (i - 1) I] (I - \hat{A})^{-2} j + \hat{A}^{i-1} E_0 + (I - \hat{A})(I - \hat{A})^{-1} H,$$  \hspace{1cm} (34)
which is valid for \( i = 1, 2, 3, \ldots \), and represents the overall state just before the switching moment \( t = iT \) in terms of the initial state \( X_0 \). For a stable closed-loop system, where the eigenvalues of matrix \( \hat{A} \) are within the unit circle, \( \lim_{t \to \infty} \hat{A}^t \) is zero and (34) is reduced to the following relationship:

\[
\lim_{t \to \infty} X^- = (I - \hat{A})^{-1}[iH + (I - \hat{A})^{-1}J].
\]

(35)

Using (35), (14) and (23), the steady-state expression for \( X(t) \) is obtained as:

\[
\lim_{t \to \infty} X(iT + t') = Q(t')i + U(t'), \quad t' \in (0, T),
\]

(36)

where

\[
Q(t') = e^{i\hat{A}t'}[I(I - \hat{A})^{-1}] + LA^{-1}B_0 - A^{-1}B_0
\]

(37)

\[
U(t') = e^{i\hat{A}t'}[I(I - \hat{A})^{-1}H - (I - \hat{A})^{-1}J] + [(e^{i\hat{A}t'} - I)A^{-1} - t']A^{-1}B_0.
\]

(38)

Since the DC gain of the equivalent plant in Fig. 2 is unity, and the DC gain from input \( v \) to state vector \( \hat{X}_r \) is \( -A^{-1}B_0 = \left[ \frac{[1, 0, \ldots, 0]^T}{V} \right] \), the DC gain of the closed-loop system from input \( r \) to the overall state \( X \), considering no signal transformation, is

\[
-A^{-1}B = \left[ \begin{array}{cc} -A^{-1}B_0 & \delta_c \\ 0_{(n-1) \times 1} & V \end{array} \right]
\]

(39)

where vector \( V \) describes the closed-loop DC gain from \( r \) to the compensator state \( X_c \), and \( \delta_c \) is the closed-loop DC gain from input \( r \) to output \( y \). Using (18), (16) and (39), and the fact that \( I^{-1} = \hat{I} \), the coefficient \( f \) in (36) can be simplified to:

\[
Q(t') = e^{i\hat{A}t'}(I - \hat{A})^{-1}i - 1 - A^{-1}B_0.
\]

(40)

Using (3), (1), (9) and (36), one can show that the plant state in steady-state can be described as:

\[
\lim_{t \to \infty} X_p(iT + t') = g_1[I_0, 0]Q(t') + A^{-1}B_0a_0 = \begin{cases} \frac{1 - g_1}{2} & \text{if } i \text{ is odd}, \\
0 & \text{if } i \text{ is even}. \end{cases}
\]

(41)

Using equality \(-A^{-1}B_0 = \left[ \frac{[1, 0, \ldots, 0]^T}{V} \right] \) and successive use of Eq. (13), the coefficient \( f \) in the steady-state solution of the plant state vector can be expressed in the following form:

\[
(1)P(t')(\delta_c - 1).
\]

(42)

which reveals that the plant state tends to a bounded value if and only if either the closed-loop DC gain \( \delta_c \) is unity, or all the elements in the time-dependent vector \( P(t') \), defined in (30), are identically zero.

Since condition (30) is almost impossible to occur, condition (29) is a necessary condition for boundedness of the plant state. The plant does not have any pole at the origin because of its unity DC gain. Hence, the only way for the closed-loop system to have a unity DC gain is that the compensator has at least one pole at the origin. Thus, a condition for boundedness of the plant state is that the compensator has at least one pole at the origin.

In the more general case of an arbitrary bounded reference signal \( x_d \), using (27), the constant vector \( H \) should be replaced by the bounded vector \( W(i - k, T) \). In this case, the last term in the right-hand side of (34) should be replaced by \( \sum_{k=0}^{i-1} \hat{A}^kW(i - k, T) \). which will not grow with \( i \), because the state matrix \( \hat{A} \) in the discrete-time LTI system (27) is stable. In this way, all of the terms which grow with \( i \), remain unchanged. Hence, the aforementioned condition about the boundedness of the plant state is not restricted to the triangular desired waveform and is valid for any arbitrary bounded reference signal \( x_d \) as well.

**Corollary 5.** Assuming unity DC gain for the plant, signal transformation parameters (1), eigenvalues of matrix \( \hat{A} \) within the unit circle, and vector \( P(t') \) defined in (30) not identically zero, a sufficient and necessary condition for the plant state to remain bounded in the hybrid control system is that the compensator has at least one pole at the origin.

The position error \( e_p := x_d - x \) is simply related to the tracking error \( e := y - y \) by the following formula:

\[
e_p := x_d - x = g_2^{-1}e = (-1)^{i-1}(r - y),
\]

for \( iT < t < iT \).

(43)

Assuming the triangular desired signal shown in Fig. 1 for \( x_d \), the conditions mentioned in **Corollary 5**, and condition (29), one can show that the profile of the error in steady-state can be expressed in the following form:

\[
\lim_{t \to \infty} e_p(t' + iT) = (1)^{i-1}C(e^{i\hat{A}t'}(I - \hat{A})^{-1}(E - I) + e^{i\hat{A}t'} - 1)A^{-2}B_0.
\]

(44)

where \( t' \in (0, T) \). Eq. (44) can be further simplified into the following form if the compensator has at least two poles at the origin.

\[
\lim_{t \to \infty} e_p(t' + iT) = (1)^{i-1}C(e^{i\hat{A}t'}(I - \hat{A})^{-1}(E - I) + e^{i\hat{A}t'} - 1)A^{-2}B_0.
\]

(45)

In this case, the profile of the steady-state error with no signal transformation is in the following form:

\[
e_p(t' + iT) = (1)^{i-1}2Ce^{i\hat{A}t'}(I - \hat{A})^{-1}(E - I)A^{-2}B_0.
\]

(46)

Eqs. (44)–(46) are useful for fast and accurate calculation of the steady-state profile of the error. Moreover, for a first-order plant, vector \( (I - 1)A^{-2}B \) vanishes and the steady-state error (45) is identically zero.

### 4. Robustness

The superiority of the signal transformation method over ordinary 1-DoF feedback systems to reduce the steady-state error was shown by simulation and experiment in Sebastian and Moheimani (2009). However, the control system becomes sensitive to disturbances and uncertainty in plant DC-gain if signal transformation blocks are incorporated around the plant, as in Fig. 1, without an appropriate inner feedback loop around the plant. In this section, we use simulations to evaluate the control performance and robustness of the signal transformation method compared with a 2-DoF feedback system. We also show how robustness of ordinary feedback systems against disturbances and uncertainties may be restored in a control system with signal transformation. To reduce the effect of high frequency components of the measurement noise at the output, the controllers are designed such that the noise transfer function, which is defined to be the closed-loop transfer function from measurement noise \( n \) to the plant output \( x \), in the feedback system shown in Fig. 3, has a limited bandwidth, which will be referred as the noise bandwidth. Such a constraint limits the tracking performance of the control system. In ordinary feedback control systems, for an acceptable tracking of a triangular reference, the fundamental frequency should be less than one third of the closed-loop bandwidth of the transfer function from the reference to the controlled output (Salapaka et al., 2002).
The nominal plant, whose frequency response is shown in Fig. 4, has a unity DC gain, a zero at $-200$ rad/s, and a pair of stable complex poles with a low damping ratio of 0.01 and angular frequency of 100 rad/s. Because of parameter uncertainty and undesirable resonance behavior of the plant, open-loop control is inapplicable and use of a feedback controller is inevitable. With the compensator in Fig. 3 being a PID controller in the form of

$$C_{pid}(s) = 9.1 + \frac{980}{s} + \frac{0.0559s}{1 + 0.0055s}.$$  (47)

the closed-loop system has an infinite gain margin, a phase margin of $82^\circ$, a bandwidth of 183 Hz, desirable stable poles, and a robust unity DC gain. We now incorporate the signal transformation method, treating the foregoing closed-loop system, excluding the pre-filter, as the plant block in Fig. 1. In this way, the noisy output $x_m$ is fed into the last signal transformation block $\Phi$ in Fig. 1. The compensator in Fig. 1 is a double integrator with the following transfer function, which is a popular controller for tracking of ramp signals (Belanger and Luyben (1997)).

$$K(s) := \frac{k_i}{s} + \frac{k_{ii}}{s^2}.$$  (48)

The state-space representation of the controller can be realized by the following matrices.

$$A_c = \begin{bmatrix} 0 & k_{ii} \\ 0 & 1 \end{bmatrix}, \quad B_c = \begin{bmatrix} k_i \\ 1 \end{bmatrix}, \quad C_c = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$  (49)

With controller gains $k_i = 500$ and $k_{ii} = 2000$, the overall closed-loop system with signal transformation blocks temporarily replaced by unity gains has an infinite gain margin, a phase margin of $62^\circ$, a noise bandwidth of 250 Hz, and the transfer function from reference to output has a bandwidth of 128 Hz (see Fig. 4).

For comparison purposes, we also construct an ordinary 2-DoF feedback control system by incorporating a pre-filter $F(s)$ in series with the input in Fig. 3. The pre-filter in the ordinary feedback system does not affect the noise bandwidth but can be adjusted to increase the bandwidth of the transfer function from reference to output. The compensator of the ordinary system in Fig. 3 is the PID controller (47) multiplied by a factor of 1.4 in order to have the same overall noise bandwidth as that of the system with signal transformation (i.e. 250 Hz, also see Fig. 4). In this way, the system has an infinite gain margin and a phase margin of $84.5^\circ$. The pre-filter for the ordinary system was selected in the following form

$$F_o(s) = \frac{1 + \frac{s}{1267.63}}{1 + \frac{s}{1267.63}}.$$  (50)

which provides a nominal forward bandwidth of 24.7 kHz for the closed-loop system. The closed-loop Bode diagrams, noise and forward transfer functions for the ordinary system are also shown in Fig. 4.

The desired triangular signal has unity amplitude $d_0 = 1$ and a half period of $T = 0.05$ s. We also apply output and input step disturbances at 0.5 s and 0.7 s, with magnitudes 1 and 10, respectively. The steady-state errors, shown in Fig. 5(a), have root-mean-square values of 0.003 and 0.012 with signal transformation and with the ordinary feedback system, respectively. The transient responses shown in Fig. 5(b) show that both systems have almost the same output disturbance rejection performance and the effect of an input disturbance is rejected faster in the system with signal transformation. Hence, with a limited noise bandwidth, the system with signal transformation provides better performances for tracking error and input disturbance compared to the ordinary 2-DoF feedback system.

With the above-mentioned controller parameters, signal transformation maintains its superiority over the 2-DoF controller when the plant DC gain is set to any value in the interval of $[0.3, +\infty]$, or the damping ratio of the plant poles is adopted from the interval of $[-2.5, 5]$, or the plant zero changes to a value in the interval of $[-400, -2]$ rad/s. Hence, with the proposed strategy, the signal transformation method can handle large parameter uncertainties including unstable plant poles.

To consider the effect of unstructural uncertainty, a pair of complex stable poles with damping ratio of 0.1 and angular frequency of 7000 rad/s was added to the nominal plant as unmodeled dynamics. For the ordinary 2-DoF feedback system, the aforementioned factor of 1.4 was changed to 1 to prevent instability and the pre-filter was changed accordingly to keep the same bandwidth of 24.7 kHz $\left(F_o = \frac{1 + s/853.75}{1 + s/853.75}\right)$. For a fair comparison, in the system with signal transformation, the PID controller was multiplied by a factor of 0.64 (without changing the other parameters) in order to keep the noise bandwidth similar to that of the ordinary 2-DoF system (183 Hz). As shown in Fig. 6, with a constraint on noise bandwidth, the signal transformation retains its superiority over the 2-DoF controller, in regard to the steady-state error and vibration suppression, even in the presence of unmodeled dynamics.

The following parameter can be considered as a measure of transient performance for the signal transformation method.

$$\gamma := |\lambda_{\max}(\hat{A})|.$$  (51)

This parameter represents the magnitude of the pole of the closed-loop discrete-time system (13), which is closer to the unit circle than the other poles and should be less than 1 for stability. Moreover, the smaller the value of $\gamma$, the faster is the decay rate of matrices $\hat{A}^{-i}$ and $\hat{A}^i$ as $i$ tends to infinity. The value of this parameter with the selected nominal plant, compensators,
and reference signal (no unmodeled dynamics) is 0.817. If we decrease the period of the desired triangular reference, the steady-state error will increase in both systems, and so will $\gamma$, which renders the system with the signal transformation unstable, while in the ordinary feedback system the stability is not affected by the period. In the above example, if we increase the fundamental frequency of the triangular reference up to 25 Hz, the system with signal transformation keeps the aforementioned superiorities over the ordinary system. If the speed of the triangular reference is increased by a factor of ten (100 Hz), the steady-state errors in both systems are almost the same. However, if the constraint on noise bandwidth is relaxed and the double integrator gains are increased to $k_i = 40000$ and $k_{ii} = 20000$, signal transformation can introduce much better steady-state performance for such a fast reference compared to the ordinary system, as shown in Fig. 7. In this case, the tracking error in the ordinary system cannot be improved by increasing the gains of the PID controller due to the unmodeled dynamics.

### 5. Sufficient conditions for steady-state improvement

In this section, we investigate conditions under which incorporation of signal transformation blocks provides tracking performance improvement. We assume that signal transformation is applied in a manner similar to the previous section, where disturbances and uncertainties are taken care of by an appropriate inner feedback loop around the original plant. For convenience, the word “plant” refers to the forward closed-loop transfer function in Fig. 3 (excluding the pre-filter), which is used as the plant block in Fig. 1. Before finding a general criterion under which the control performance is improved by the signal transformation, it is helpful to consider effects of plant zeros by simulation.

### 5.1. Effect of plant zeros

We start with a third order plant with unity DC gain in Fig. 1, poles at $-100, -10000, -1000$, and no zeros. A compensator in the form of (48) with unity coefficients reduces the open-loop bandwidth of 16 Hz to a closed-loop bandwidth of 0.29 Hz with phase margin of 51° and gain margin of 60 dB. In this case, incorporation of signal transformation improves the steady-state tracking error for triangular references with fundamental frequencies of up to 5 Hz (16 times more than the closed-loop bandwidth). If we include two zeros at $-150$ and $-1500$ rad/s into the plant while keeping its DC gain and compensator unchanged, we still find the signal transformation method will introduce similar improvements. The improvements are also repeated even if either of the plant zeros has a sign change, which renders the plant a non-minimum-phase system. However, if we move the plant zeros to $-15$ and $-150$ rad/s, where the dominant zeros are closer to the origin than the dominant open-loop poles, the signal transformation method produces large errors at the initial moments of each half period, as shown in Fig. 8(a) and (b) for a unity amplitude triangular reference. In this case, where the closed-loop bandwidth is 0.27 Hz with unity compensator gains, incorporation of signal transformation cannot improve the steady-state tracking performance for a faster triangular reference with fundamental frequency of 5 Hz, as shown in Fig. 8(c) and (d). In the foregoing case, the ordinary feedback system has an unlimited gain margin and a safe phase margin of 56°. However, its response to a unit step disturbance applied at the plant input has an undesirable overshoot of 35, which is mainly due to the dominant zero at $-15$ rad/s.
considers a minimum value for the half – (63) \( - (63) \) \( 4 \) zeros at \(- 15\) and \(-150\) rad/s. (a), (b) For a 0.5 Hz reference. (c), (d) For a 5 Hz reference.

5.2. Second order plants with double integrator compensator

From the simulation results in Section 4 we observed that the signal transformation can introduce steady-state improvement for tracking of a triangular reference signal if the ordinary feedback system meets at least some mild stability conditions. In this section, we are going to show that satisfaction of these mild stability margins is almost a sufficient condition for the signal transformation method to introduce steady-state improvement for second-order plants. The second-order plant is assumed to be stable and to have a unity DC gain, no zeros, and complex poles in the following form:

\[ R(- \cos \theta \pm i \sin \theta), \quad 0 < R, \quad 0 < \theta < \frac{\pi}{2}. \] (52)

To simplify the calculation of range of compensator parameters meeting the mild stability margins, we restrict the compensator structure into the double integrator form (48), which can be represented in a normalized form w.r.t. the magnitude of plant poles as:

\[ K(s) = \bar{k}_i \frac{1 + \frac{s}{R}}{s^2}, \] (53)

where

\[ \bar{s} := \frac{s}{R}, \quad \bar{k}_i := \frac{k_i}{R}, \quad \bar{z} := \frac{k_i}{\bar{k}_i R}. \] (54)

If the compensator is to provide the needed mild stability margins for the ordinary feedback system, then the system needs to be stable. The closed-loop transfer functions of the ordinary feedback system from the reference input to the plant output, in terms of the normalized Laplace variable \( \bar{s} \), can be written in the following form.

\[ T(\bar{s}) \approx \frac{\bar{k}_i (1 + \frac{1}{\bar{s}^2})}{\bar{s}^4 + 2 \cos \theta \bar{s}^3 + \bar{s}^2 + \frac{2 \bar{k}_i}{\bar{k}_i R} \bar{s} + \bar{k}_i}. \] (55)

The first column of the Routh–Hurwitz table associated with the denominator of \( T(\bar{s}) \) can be described as:

\[ Z(\theta, \bar{k}_i, \bar{z}) := \begin{bmatrix} 1 & 2 \cos \theta & 1 - \frac{\bar{k}_i}{\bar{z} \cos \theta} & \frac{\bar{k}_i}{\bar{z}} - \frac{2 \bar{k}_i \cos \theta}{1 - \frac{\bar{k}_i}{\bar{z} \cos \theta}} \end{bmatrix}^T. \] (56)

Since the first element of \( Z \) is positive, the ordinary feedback system is stable if and only if all of the other elements of \( Z \) are positive. The second element is also positive because \( \theta \) belongs to range \((0, \frac{\pi}{2})\). The last element shows that \( \bar{k}_i \) needs to be positive. Having \( \bar{k}_i \) and \( \cos \theta \) positive, the fourth element of \( Z \) shows that \( \bar{z} \) should be positive. Using the positiveness of the third element in the fourth element, one can reach the inequality \( \bar{k}_i < \bar{z} \cos \theta (1 - \bar{z} \cos \theta) \). Since \( \bar{k}_i \) is positive, the foregoing inequality leads to the following sufficient and necessary conditions for stability of the ordinary feedback system.

\[ 0 < \bar{z} < \frac{1}{2 \cos \theta} \] (57)

\[ 0 < \bar{k}_i < \bar{z} \cos \theta (1 - 2 \bar{z} \cos \theta). \] (58)

The above conditions clearly show the range of normalized compensator parameters over which the ordinary feedback system is stable. One can also obtain a closed-form relationship for the gain margin of the ordinary feedback system in the following form:

\[ GM = 20 \log_{10} \left[ \frac{2 \bar{z} \cos \theta (1 - 2 \bar{z} \cos \theta)}{\bar{k}_i} \right], \quad (\text{dB}). \] (59)

We assume that the compensator satisfies a minimum gain margin of \( GM_{\text{min}} = 10 \) dB, which will replace the stability condition (58) with the following more restrictive condition,

\[ 0 < \bar{k}_i < 0.6325 \bar{z} \cos \theta (1 - 2 \bar{z} \cos \theta). \] (60)

The other stability margins that we also consider for the closed-loop ordinary feedback system are as follows:

- Phase margin > \( 10^\circ \) (61)
- Bandwidth of \( T(s) \) (62)
- Bandwidth of plant
- \( \max |T(i \omega)| < 6 \) dB

where condition (62) is to prevent amplification of measurement noise, and condition (63) is to limit the overshoot in the step response. Because of lack of analytic expressions for phase margin, control bandwidth, and maximum magnitude of \( T(i \omega) \), we will use numerical methods to determine how conditions (61)–(63) impose further restrictions on the range of normalized compensator parameters \( \bar{k}_i \) and \( \bar{z} \). Because the final range of compensator parameters meeting conditions (57)–(63) is a subset of the range specified by (57) and (60), these two conditions help the numerical method to limit the search ranges and obtain more exact solutions for the final range. Because we want to draw some conclusions about the steady-state response of the hybrid control system to the triangular reference, which includes the signal transformation, we also consider the following conditions before obtaining the range of compensator parameters numerically.

\[ T = \frac{10}{R \cos \theta} \] (64)

\[ \gamma < 0.9 \] (65)

where condition (64) considers a minimum value for the half period of the desired triangular waveform, which was selected ten times more than the time constant of the open-loop plant, and condition (65) is for stability of the hybrid system (see (51)). The representations of plant and compensator transfer functions
in terms of $\cos \theta$ and the normalized variables $\bar{s}$, $\bar{z}$, and $\bar{k}_i$ do not depend on plant pole magnitude $R$. One can also show that the phase margin, the bandwidth ratio defined in the left-hand side of condition (62), the peak magnitude of $T(\omega_i)$, and $\gamma$ just depend on the normalized variables and do not depend on $R$. Moreover, the profiles of the steady-state error in both of the ordinary and hybrid feedback systems, presented by Eq. (46) and (45), respectively, depend just on the normalized time variable $T := \frac{t - \bar{T}}{\bar{z}}$, $\cos \theta$, $\bar{z}$, and $\bar{k}_i$, and do not depend on $R$. The contours in Fig. 9 show the range of normalized compensator parameters that satisfy conditions (57) and (60)–(65) for seven fixed values of plant pole angles $\theta$ in the range of $(0^\circ, 90^\circ)$, which shows that the region of acceptable compensator parameters becomes smaller as the plant poles approach the imaginary axis. With the compensator parameters within the acceptable ranges, the rms-value of the steady-state error in the hybrid system was found to be at least six times less than that of the ordinary feedback system, where the worst case occurs for very small compensator parameters and $\theta = 37^\circ$. Since the acceptable design domains shown in Fig. 9 for the double-integrator compensator is valid for any magnitude of the plant poles, the mentioned steady-state improvement is valid for any second-order unity DC gain plant with stable complex poles and no zeros.

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References


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