Optimal quadratic guaranteed cost control of a class of uncertain time-delay systems

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Abstract: The design of robust state feedback controllers for a class of uncertain linear time-delay systems with norm-bounded uncertainty is presented. The state feedback results extend previous results on quadratic guaranteed cost control to the case of uncertain time-delay systems. This is done by the authors' definition of quadratic stability for uncertain time-delay systems with norm bounded uncertainty. It is shown that the state feedback controller can be constructed via the solution of a parameter dependent Riccati equation.

1 Introduction

Delays often occur in the transmission of material or information between different parts of a system. Communication systems, transmission systems, chemical processing systems, metallurgical processing systems, environmental systems and power systems are examples of time-delay systems. Considerable research has been done on various aspects of non-uncertain time-delay systems in the past 30 years; for example, see [1] and references therein. However, little attention has been paid towards uncertain time-delay systems. In this paper, we extend the linear quadratic regulator to the case in which the underlying system contains time delays and is uncertain.

Our results on designing linear quadratic regulators for the time-delay systems with parameter uncertainty fall within the area of 'robust control theory', which is concerned with the problem of designing feedback controllers for uncertain plants. The controller is to be constructed such that the closed-loop system is stable and an adequate level of performance for all admissible values of uncertainty is guaranteed. Our approach to guaranteeing closed-loop stability is to use a fixed quadratic Lyapunov function for the closed-loop system. This approach leads us to a definition of quadratic stability for uncertain linear time-delay systems which is closely related to the definition of quadratic stability for uncertain linear systems [2].

In this paper, we consider the problem of designing a robust state feedback controller that makes the closed-loop system quadratically stable and guarantees an adequate level of performance. The performance index considered here is assumed to be an integral quadratic cost function such as occurs in the linear quadratic regulator problem. Our approach to this problem is the so-called 'guaranteed cost control' approach in which a fixed quadratic Lyapunov function is used to establish an upper bound on the cost function [3-6]. This motivates our definition of 'quadratic guaranteed cost control', which is closely related to the definition given in [6]. Our main result shows that an optimal state feedback controller may be constructed via solving a parameter dependent Riccati equation. Moreover, if the time-delay terms vanish, our results reduce to those of [6].

2 Analysis of robust performance

We consider a continuous-time autonomous uncertain time-delay system of the form

$$x(t) = \left( A + \tilde{D} \Delta(t) \tilde{E} \right) x(t) + \tilde{F}_1 x(t-\tau_1) + \cdots + \tilde{F}_r x(t-\tau_r)$$

(1)

where $x(t) \in \mathbb{R}^n$ is the state, and $\Delta(t) \in \mathbb{R}^{m \times m}$ is a matrix of uncertain parameters satisfying the bound $\Delta(t)' \Delta(t) \leq I$. Here, $\tau_1, \ldots, \tau_r$ are time delays. The initial condition is specified as $(x(0), x(s)) = (x_0, \phi(s))$, where $\phi(\cdot) \in \mathcal{D}^{[-\tau, 0]}$ and $\tau = \max(\tau_1, \ldots, \tau_r)$. We also assume that each $\tilde{F}_i$ has been factored as $\tilde{F}_i = G_i H_i$, where $H_i \in \mathbb{R}^{l_i \times m}$ and $l_i = \text{rank}(\tilde{F}_i)$.

Associated with this system is the cost function

$$J = \int_0^\infty x(t)' Q x(t) dt$$

(2)

where $Q > 0$.

In this paper, we use a quadratic Lyapunov function to give an upper bound on the cost function (eqn. 2).

Definition 1: The uncertain delay system (eqn. 1) is said to be quadratically stable if there exists a symmetric positive definite matrix $P > 0$ and positive numbers $\delta_1, \ldots, \delta_r$ such that

$$\left( \bar{A} + \bar{D} \Delta \bar{E} \right)' P + P \left( \bar{A} + \bar{D} \Delta \bar{E} \right) + \sum_{i=1}^r \delta_i \bar{H}_i' \bar{H}_i$$

$$+ P \left( \sum_{i=1}^r \frac{1}{\delta_i} \bar{G}_i \bar{G}_i' \right) P < 0$$

(3)

for all $\Delta: \Delta' \Delta \leq I$.

This definition extends the definition of quadratic stability to the case of uncertain linear time-delay systems of the form of eqn. 1. Note that this definition...
is independent of the time delays $\tau_1, ..., \tau_r$. Moreover, if the time delay terms are not present, i.e. $\bar{H}_i = 0$ and $\bar{G}_i = 0$ for $i = 1, ..., r$, then the above definition reduces to the standard definition of quadratic stability for uncertain linear systems with norm bounded uncertainty [7].

A motivation for this definition of quadratic stability is the following theorem.

**Theorem 1:** Consider the system of eqn. 1 and the Lyapunov function

$$V(x(t)) = x(t)' \bar{P} x(t) + \sum_{i=1}^{r} \int_0^{\tau_i} x(t-s)' (\delta_i \bar{H}_i \bar{H}_i) x(t-s) ds $$

If the system (eqn. 1) is quadratically stable, then there exists a Lyapunov function of the form of eqn. 4 such that $V < 0$, and hence $x(t) \to 0$ as $t \to \infty$.

Conversely, if there exist $\bar{P} > 0$ and $\delta_i > 0$ such that $V(x(t)) < 0$, then the system is quadratically stable.

**Proof:** The theorem can be proved by differentiating $V$ with respect to time and using the Schur complement of eqn. 3. It should be noted that in [8] and [9] similar Lyapunov functions are used to stabilise uncertain systems. However, [8] and [9] do not consider problems of guaranteed cost control. Also, a similar Lyapunov function is used in [10] to stabilise time-delay systems using an approach based on LMIs. However, [10] does not address the guaranteed cost problem.

**Definition 2:** A positive definite matrix $\bar{P}$ is said to be a quadratic cost matrix for the eqn. 1 system and cost function (eqn. 2) if there exist constant numbers $\delta_1 > 0$, ..., $\delta_r > 0$ such that

$$ (\bar{A} + \bar{D} \Delta \bar{E})' \bar{P} + \bar{P} (\bar{A} + \bar{D} \Delta \bar{E}) + \sum_{i=1}^{r} \delta_i \bar{H}_i \bar{H}_i' \bar{P} + \bar{Q} < 0 $$

**Theorem 2:** Suppose $\bar{P} > 0$ is a quadratic cost matrix for the uncertain eqn. 1 system and cost function (eqn. 2). Then the system is quadratically stable and the cost function satisfies the bound

$$ J \leq x_0' \bar{P} x_0 + \sum_{i=1}^{r} \int_{-\tau_i}^{0} x(s)' (\delta_i \bar{H}_i \bar{H}_i) x(s) ds $$

Conversely, if the eqn. 1 system is quadratically stable, then there will exist a quadratic cost matrix for this system and cost function (eqn. 2). Then the system is quadratically stable and the cost function satisfies the bound

$$ J \leq x_0' \bar{P} x_0 + \sum_{i=1}^{r} \int_{-\tau_i}^{0} x(s)' (\delta_i \bar{H}_i \bar{H}_i) x(s) ds $$

**Proof:** Suppose $\bar{P} > 0$ is a quadratic cost matrix for the uncertain eqn. 1 system and cost function (eqn. 2); it follows from eqn. 5 that there exist $\delta_1 > 0$, ..., $\delta_r > 0$ such that

$$ (\bar{A} + \bar{D} \Delta \bar{E})' \bar{P} + \bar{P} (\bar{A} + \bar{D} \Delta \bar{E}) + \sum_{i=1}^{r} \delta_i \bar{H}_i \bar{H}_i' \bar{P} + \bar{Q} < 0 $$

for all $\Delta: \Delta \Delta \preceq I$. Since $\bar{Q} > 0$, this implies that

$$ (\bar{A} + \bar{D} \Delta \bar{E})' \bar{P} + \bar{P} (\bar{A} + \bar{D} \Delta \bar{E}) + \sum_{i=1}^{r} \delta_i \bar{H}_i \bar{H}_i' \bar{P} < 0 $$

for all $\Delta: \Delta \Delta \preceq I$. Since $\bar{Q} > 0$, this implies that

$$ (\bar{A} + \bar{D} \Delta \bar{E})' \bar{P} + \bar{P} (\bar{A} + \bar{D} \Delta \bar{E}) + \sum_{i=1}^{r} \delta_i \bar{H}_i \bar{H}_i' \bar{P} < 0 $$

for all $\Delta: \Delta \Delta \preceq I$. Therefore the eqn. 1 system is quadratically stable.

Now consider the Lyapunov function (eqn. 4). Differentiating $V$ with respect to time, we obtain

$$ \frac{d}{dt} V(x(t)) = [x(t)' x(t - \tau_1)' \bar{H}_1 \cdots x(t - \tau_r)' \bar{H}_r'] $$

$$ \times \left[ \begin{array}{c} \bar{A} + \bar{D} \Delta \bar{E} \bar{P} \\ + \bar{P} (\bar{A} + \bar{D} \Delta \bar{E}) \\ \sum_{i=1}^{r} \delta_i \bar{H}_i \bar{H}_i' \\ \bar{G}_1' \bar{P} - \delta_1 I \\ \vdots \\ \bar{G}_r' \bar{P} - \delta_r I \end{array} \right] $$

$$ < [x(t)' x(t - \tau_1)' \bar{H}_1 \cdots x(t - \tau_r)' \bar{H}_r'] $$

$$ \times \left[ \begin{array}{c} \bar{Q} \\ 0 \\ \vdots \\ 0 \end{array} \right] \left[ \begin{array}{c} x(t) \\ 0 \\ \vdots \\ 0 \end{array} \right] $$

$$ = -x(t)' \bar{Q} x(t) $$

Integrating both sides of the above inequality from 0 to $\infty$ we obtain

$$ \int_0^{\infty} x(t)' \bar{Q} x(t) dt \leq V(x(0)) - V(x(\infty)) $$

Since the quadratic stability of the system has already been established, we conclude that $V(x(t)) \to 0$ as $t \to \infty$. Hence,

$$ J \leq x_0' \bar{P} x_0 + \sum_{i=1}^{r} \int_{-\tau_i}^{0} x(s)' (\delta_i \bar{H}_i \bar{H}_i) x(s) ds $$

To prove the second part of the theorem, suppose that the eqn. 1 system is quadratically stable. This implies that there exist a matrix $\bar{P} > 0$ and constants $\delta_1 > 0$ such that

$$ (\bar{A} + \bar{D} \Delta \bar{E})' \bar{P} + \bar{P} (\bar{A} + \bar{D} \Delta \bar{E}) + \sum_{i=1}^{r} \delta_i \bar{H}_i \bar{H}_i' \bar{P} < 0 $$

for all matrices $\Delta$ satisfying $\Delta \Delta \preceq I$. Therefore we can find an $\epsilon > 0$ such that the following inequality is satisfied for all $\Delta$ such that $|\Delta| \leq 1$:

$$ \bar{Q} + \frac{1}{\epsilon} \left\{ (\bar{A} + \bar{D} \Delta \bar{E})' \bar{P} + \bar{P} (\bar{A} + \bar{D} \Delta \bar{E}) + \sum_{i=1}^{r} \delta_i \bar{H}_i \bar{H}_i' \bar{P} \right\} \bar{P} < 0 $$

This means that there exist constants $\delta_i = \epsilon \delta_i$ such that the matrix $\bar{P} = (1/\epsilon) \bar{P}$ is a quadratic cost matrix for the eqn. 2 system.
The following theorem gives a characterisation of all quadratic cost matrices in terms of a quadratic matrix inequality.

**Theorem 3:** A matrix \( \hat{P} > 0 \) is a quadratic cost matrix for the eqn. 1 system and cost function (eqn. 2) if and only if there exist parameters \( \delta_i > 0 \) and \( \epsilon > 0 \) such that

\[
\hat{A}' \hat{P} + \hat{P} \hat{A} + \hat{P} \left( \epsilon \bar{D} \bar{D}' + \sum_{i=1}^{r} \frac{1}{\delta_i} \hat{G}_i \hat{G}_i' \right) \hat{P} \\
+ \frac{1}{\epsilon} \bar{E}' \bar{E} + \sum_{i=1}^{r} \delta_i \hat{H}_i' \hat{H}_i + \hat{Q} < 0 \quad (7)
\]

**Proof:**

(i) **Sufficiency:** Suppose \( \hat{P} \) satisfies eqn. 7 for some \( \epsilon > 0 \) and \( \delta_i > 0 \). For any \( \Delta \) satisfying \( \Delta \Delta \leq I \) we have

\[
\left( \hat{A} + \bar{D} \Delta \bar{E} \right)' \hat{P} + \hat{P} \left( \hat{A} + \bar{D} \Delta \bar{E} \right) + \hat{P} \left( \sum_{i=1}^{r} \frac{1}{\delta_i} \hat{G}_i \hat{G}_i' \right) \hat{P} \\
+ \frac{1}{\epsilon} \bar{E}' \bar{E} + \sum_{i=1}^{r} \delta_i \hat{H}_i' \hat{H}_i + \hat{Q} < 0
\]

This is implied from eqn. 7 and claim 1 of [7].

(ii) **Necessity:** Suppose that \( \hat{P} > 0 \) is a quadratic cost matrix for the eqn. 1 system and cost function (eqn. 2). eqn. 5 implies that

\[
\left[ \begin{array}{cccc}
\hat{A} + \bar{D} \Delta \bar{E} & \hat{P} \left( \hat{A} + \bar{D} \Delta \bar{E} \right) & \ldots & \hat{P} \hat{G}_r \\
\hat{P}' \hat{A} + \hat{P} \left( \hat{A} + \bar{D} \Delta \bar{E} \right) & \hat{P} \hat{G}_1 & \ldots & \hat{P} \hat{G}_r \\
\sum_{i=1}^{r} \delta_i \hat{H}_i' \hat{H}_i + \hat{Q} & \hat{G}_1 \hat{P} & \ldots & 0 \\
\hat{G}_r \hat{P} & 0 & \ldots & -\delta_r I
\end{array} \right] < 0
\]

for all matrices \( \Delta: \Delta \Delta \leq I \). That is,

\[
x' \left( \begin{array}{c}
\hat{A} + \bar{D} \Delta \bar{E} \\
\hat{P} \left( \hat{A} + \bar{D} \Delta \bar{E} \right) \\
\sum_{i=1}^{r} \delta_i \hat{H}_i' \hat{H}_i + \hat{Q} \\
\hat{G}_r \hat{P}
\end{array} \right) x \\
+ 2 \hat{x}' \hat{P} \left( \sum_{i=1}^{r} \hat{G}_i \hat{x}_{r_i} \right) - \sum_{i=1}^{r} x_{r_i}' \left( \delta_i I \right) x_{r_i} < 0
\]

for all \( x \neq 0 \), \( x_r \neq 0 \) and \( \Delta: \Delta \Delta \leq I \). Now, as in lemma 3.1 and observation 3.1 of [7], this implies that

\[
x' \left[ \hat{A}' \hat{P} + \hat{A} \hat{P} + \hat{Q} + \sum_{i=1}^{r} \delta_i \hat{H}_i' \hat{H}_i \right] x - \sum_{i=1}^{r} x_{r_i}' \left( \delta_i I \right) x_{r_i} < 0
\]

for all \( x \neq 0 \), \( x_r \neq 0 \) and \( \Delta: \Delta \Delta \leq I \). Hence,

\[
\left[ \begin{array}{cccc}
\hat{A}' \hat{P} + \hat{P} \hat{A} & \hat{G}_1 & \ldots & \hat{G}_r \\
\hat{P}' \hat{A} + \hat{P} \hat{A} & 0 & \ldots & 0 \\
\sum_{i=1}^{r} \delta_i \hat{H}_i' \hat{H}_i + \hat{Q} & \hat{G}_1 \hat{P} & \ldots & 0 \\
\hat{G}_r \hat{P} & 0 & \ldots & -\delta_r I
\end{array} \right] < 0
\]

and

\[
\left[ \begin{array}{cccc}
\hat{A}' \hat{P} + \hat{P} \hat{A} & \hat{G}_1 & \ldots & \hat{G}_r \\
\hat{P}' \hat{A} + \hat{P} \hat{A} & 0 & \ldots & 0 \\
\sum_{i=1}^{r} \delta_i \hat{H}_i' \hat{H}_i + \hat{Q} & \hat{G}_1 \hat{P} & \ldots & 0 \\
\hat{G}_r \hat{P} & 0 & \ldots & -\delta_r I
\end{array} \right] < 0
\]

for all \( x \neq 0 \), \( x_r \neq 0 \). Hence, theorem 4.7 of [11] implies that there exists a parameter \( \epsilon > 0 \) such that

\[
\left[ \begin{array}{cccc}
\hat{A}' \hat{P} + \hat{P} \hat{A} & \hat{G}_1 & \ldots & \hat{G}_r \\
\hat{P}' \hat{A} + \hat{P} \hat{A} & 0 & \ldots & 0 \\
\sum_{i=1}^{r} \delta_i \hat{H}_i' \hat{H}_i + \hat{Q} & \hat{G}_1 \hat{P} & \ldots & 0 \\
\hat{G}_r \hat{P} & 0 & \ldots & -\delta_r I
\end{array} \right] < 0
\]

which proves the second part of the theorem.

In the sequel, we will need the strict bounded real lemma [12]. Therefore, for the sake of completeness, we present it here. However, we first define the stabilising solution of a Riccati equation.

**Definition 3:** Consider an algebraic Riccati equation of the form

\[
A'X + XA - XMX + N = 0 \quad (8)
\]

A symmetric matrix \( X \) which satisfies this Riccati equation is said to be a stabilising solution if \( A - MX \) is stable.

**Theorem 4** [12] (The strict bounded real lemma with nonminimal realisations):

The following statements are equivalent:
1. \( A \) is stable and \( \|C(sI - A)^{-1}B\| < 1 \).
2. There exists a matrix \( P > 0 \) such that

\[
A'P + PA + PBB'P + C'C < 0
\]

3. The Riccati equation

\[
A'P + PA + PBB'P + C'C = 0
\]

has a stabilising solution \( P \geq 0 \). Furthermore, if these
Considering the strict bounded real lemma, it is possible to state the conditions of theorem 3 in terms of an $H_\infty$ norm bound condition. Theorem 4 implies that there exists a $P > 0$ such that eqn. 7 is satisfied if and only if the following conditions hold:

1. $\dot{A}$ is a stability matrix.
2. The following $H_\infty$ norm bound is satisfied:

$$
\begin{bmatrix}
\frac{1}{\sqrt{\varepsilon}} \tilde{E} \\
\sqrt{\varepsilon_1} \tilde{H}_1 \\
\vdots \\
\sqrt{\varepsilon_r} \tilde{H}_r \\
\tilde{Q}^T
\end{bmatrix}
(sI - \tilde{A})^{-1}
\begin{bmatrix}
\frac{1}{\sqrt{\varepsilon}} \tilde{G}_1 \\
\sqrt{\varepsilon_1} \tilde{G}_1 \\
\vdots \\
\sqrt{\varepsilon_r} \tilde{G}_r \\
\tilde{Q}^T
\end{bmatrix}
< 1
\infty
$$

(9)

Therefore, the set of values of $\varepsilon$, and $\delta_i$ for which the quadratic matrix eqn. 7 has a positive definite solution, can be found by a search over those $\varepsilon > 0$, and $\delta_i > 0$ such that the eqn. 9 is satisfied. Furthermore, theorem 4 implies that, for any such $\varepsilon$ and $\delta_i$, the following Riccati equation has a stabilising solution $\dot{P} \equiv 0$:

$$\begin{align*}
\dot{A}'\dot{P} + \dot{P}A &+ P(\varepsilon\tilde{D}\tilde{D}' + \sum_{i=1}^{r} \frac{1}{\delta_i} \tilde{G}_i \tilde{G}_i^T)\dot{P} \\
+ \frac{1}{\varepsilon} \tilde{E}'\tilde{E} + \sum_{i=1}^{r} \delta_i \tilde{H}_i' \tilde{H}_i + \tilde{Q} &= = 0
\end{align*}
$$

3 Controller synthesis

In this Section we consider the problem of optimal guaranteed cost control via state feedback for a class of uncertain time-delay systems. The uncertain time-delay systems under consideration are described by the state equations

$$\begin{align*}
\dot{x}(t) &= (A + D_\Delta(t)E_1)x(t) + F_1x(t - \tau_1) + \cdots \\
&+ F_r x(t - \tau_r) + (B + D_\Delta(t)E_2)u(t)
\end{align*}
$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $\tau_1 > 0, \ldots, \tau_r > 0$ are time delays and $\Delta(t) \in \mathbb{R}^{m \times q}$ is a time-varying matrix of uncertain parameters satisfying the bound $\Delta(t)/\Delta(t) \leq I$.

Associated with this system is the cost function

$$J = \int_0^\infty [x(t)'Qx(t) + u(t)'Ru(t)]dt
$$

(11)

Definition 4: A control law $u(t) = Kx(t)$ is said to define a quadratic guaranteed cost control with cost associated matrix $P > 0$ for the eqn. 10 system and cost function (eqn. 11) if there exists a $\delta > 0$ such that

$$\begin{align*}
(A + BK + D_\Delta(E_1 + E_2K)')'P \\
+ P(A + BK + D_\Delta(E_1 + E_2K))'P \\
+ P \left(\sum_{i=1}^{r} \frac{1}{\delta_i} G_i G_i^T\right) P + K'RK + \sum_{i=1}^{r} \delta_i H_i' H_i + Q &< 0
\end{align*}
$$

(12)

for all $\Delta$: $\Delta'\Delta \leq I$.

We will need the following lemma to prove the main result of this paper.

Lemma 1: Let $Q_1 = Q_1' > 0$, $Q_2 = Q_2' > 0$ and $W = W' > 0$ be given matrices and suppose the Riccati equation

$$\begin{align*}
A'P + PA + PWP + Q_1 &= = 0
\end{align*}
$$

has a positive definite solution $P > 0$. Furthermore, suppose $Q_2 < Q_1$. Then the Riccati equation

$$\begin{align*}
A'S + SA + SWS + Q_2 &= = 0
\end{align*}
$$

(14)

has a positive definite solution $S$ such that $S < P$.

Proof: Suppose Riccati eqn. 13 has a positive definite solution $P > 0$. Since $Q_2 < Q_1$, there exists a matrix $M > 0$ such that $Q_2 + M = Q_1$. Therefore, the following inequality is true:

$$\begin{align*}
A'P + PA + PWP + Q_2 &= = 0
\end{align*}
$$

(15)

Theorem 4 and eqn. 15 imply that the Riccati equation (eqn. 14) has a stabilising solution $S \geq 0$ such that $S < P$. Since $Q_2 > 0$, we conclude that $S > 0$.

The following theorem is the main result of this paper. It shows that the problem of finding a quadratic guaranteed cost controller for the uncertain system (eqn. 10) and the cost function (eqn. 11) can be solved via solving a parameter dependent Riccati equation.

Theorem 5: Suppose there exist constants $\varepsilon > 0$ and $\delta_1 > 0, \ldots, \delta_r > 0$ such that the Riccati equation

$$\begin{align*}
(A - B(\varepsilon R + E_2')^{-1} E_2 E_1)P \\
+ P(A - B(\varepsilon R + E_2')^{-1} E_2 E_1) \\
+ P(\varepsilon D D' + \sum_{i=1}^{r} \frac{1}{\delta_i} G_i G_i^T) P - \varepsilon PB(\varepsilon R + E_2')^{-1} B'P \\
+ \frac{1}{\varepsilon} E_1' (I - E_2(\varepsilon R + E_2')^{-1} E_2') E_1 + \sum_{i=1}^{r} \delta_i H_i' H_i + Q \\
= = 0
\end{align*}
$$

(16)

has a stabilising solution $P > 0$, and consider the control law

$$u(t) = -(\varepsilon R + E_2')^{-1}(\varepsilon B'P + E_2')x(t)
$$

That, given any $\sigma > 0$, there exists a matrix $P > 0$ such that $P < P < P + \sigma$ and eqn. 17 is a quadratic guaranteed cost control for the eqn. 10 system with cost matrix $P$.

Conversely, given any quadratic guaranteed cost control with cost matrix $P > 0$, there exist $\delta > 0$ and $\varepsilon > 0$ such that Riccati equation (eqn. 16) has a stabilising solution $P^* > 0$, where $P^* < P$.

Proof: To establish the first part of the theorem, let the control law $u(t)$ be defined as in eqn. 17. If we define $A = A + BK$ and $\tilde{E} = E_1 + E_2K$, then it is straightforward but tedious to show that eqn. 16 is equivalent to

$$\begin{align*}
A'P + PA + PWP + Q_1 &= = 0
\end{align*}
$$

The strict bounded real lemma (theorem 4) implies that there exists a matrix $P > 0$ such that

$$\begin{align*}
A'\tilde{P} + \tilde{P}A + \tilde{P}(E D D' + \sum_{i=1}^{r} \frac{1}{\delta_i} G_i G_i^T) \tilde{P} \\
+ \frac{1}{\varepsilon} \tilde{E}' \tilde{E} + Q + K'RK + \sum_{i=1}^{r} \delta_i H_i' H_i + Q &< 0
\end{align*}
$$

Therefore, it can be argued that a matrix $M > 0$ can be
found such that
\[ A^T \hat{P} + \hat{P} A + \hat{P} \left( \varepsilon D D' + \sum_{i=1}^{r} \frac{1}{\delta_i} G_i G_i' \right) \hat{P} \]
\[ + \frac{1}{\varepsilon} E' E + Q + K' R K + \sum_{i=1}^{r} \delta_i H_i' H_i + \mu M = 0 \] (18)

Hence, given any \( \mu \) such that \( 0 < \mu < 1 \), it follows from eqn. 18 and lemma 1 that the Riccati equation
\[ A' P_{i\mu} + P_{i\mu} A + P_{i\mu} \left( \varepsilon D D' + \sum_{i=1}^{r} \frac{1}{\delta_i} G_i G_i' \right) P_{i\mu} \]
\[ + \frac{1}{\varepsilon} E' E + Q + K' R K + \sum_{i=1}^{r} \delta_i H_i' H_i + \mu M = 0 \]

has a positive-definite solution \( P_{i\mu} > 0 \). Furthermore, \( P_{i\mu} > P_i \). However, in the limit as \( \mu \to 0, P_{i\mu} \to P_i \). Hence, given any \( \alpha > 0, a > 0 \) can be found such that, if we let \( P = P_{i\mu} \), then \( P < P < P + \alpha I \). This completes the proof of the first part of the theorem.

To prove the second part of the theorem, assume that \( u(t) = K x(t) \) is a quadratic guaranteed cost controller with cost matrix \( \hat{P} \). It follows from theorem 3 that there exist \( \varepsilon > 0 \) and \( \delta > 0 \) such that
\[ A^T \hat{P} + \hat{P} A + \hat{P} \left( \varepsilon D D' + \sum_{i=1}^{r} \frac{1}{\delta_i} G_i G_i' \right) \hat{P} \]
\[ + \frac{1}{\varepsilon} E' E + Q + K' R K + \sum_{i=1}^{r} \delta_i H_i' H_i + \varepsilon K' R K + \frac{1}{\varepsilon} E_1 + E_2 K' \]
\[ + \left( \frac{1}{\varepsilon} E_1 + E_2 K' \right)' \left( \frac{1}{\varepsilon} E_1 + E_2 K' \right) < 0 \]

From this, it follows that the matrix \( \hat{P} > 0 \) satisfies the inequality
\[ \left[ A + \sqrt{\varepsilon} B K \right] \hat{P} + \hat{P} \left[ A + \sqrt{\varepsilon} B K \right] + \hat{P} W W' \hat{P} \]
\[ + Q + \sum_{i=1}^{r} \delta_i H_i' H_i + \varepsilon K' R K \]
\[ + \left( \frac{1}{\varepsilon} E_1 + E_2 K \right)' \left( \frac{1}{\varepsilon} E_1 + E_2 K \right) < 0 \]

where \( K = 1/\varepsilon \) and \( W = [v D 1/\sqrt{\delta_1 G_1} ... 1/\sqrt{\delta_r G_r}] \). Now we can apply lemma 3.1 of [6] to this inequality by considering a state feedback \( H \) control problem defined by the system
\[ \dot{x} = A x + \left[ \sqrt{\varepsilon} D \frac{1}{\sqrt{\delta_1}} G_1 \ldots \frac{1}{\sqrt{\varepsilon}} G_r \right] w + \sqrt{\varepsilon} B u \]
\[ z = \left[ \begin{array}{c} \sqrt{\varepsilon} E_1 \\ \frac{1}{\sqrt{\delta_1}} H_1 + \frac{Q}{2} \end{array} \right] x + \left[ \begin{array}{c} E_2 \\ \frac{1}{\sqrt{\varepsilon}} R_1 \end{array} \right] u \]

(20)

It is straightforward to verify that this system satisfies assumptions of lemma 3.1 of [6]. Now suppose that the state feedback control \( u(t) = K x(t) \) is applied to this system; it follows that the Riccati equation \( A' P_i + P_i A + P_i \left( \varepsilon D D' + \sum_{i=1}^{r} \frac{1}{\delta_i} G_i G_i' \right) P_i \]
\[ + \frac{1}{\varepsilon} E' E + Q + K' R K + \sum_{i=1}^{r} \delta_i H_i' H_i + \mu M = 0 \] (16)

minimising the bound (eqn. 6) which is a function of the time delays \( \tau_1, \ldots, \tau_r \).

Theorem 5 gives a characterisation of all quadratic guaranteed cost controllers in terms of a parameter dependent algebraic Riccati equation. Each quadratic guaranteed cost controller guarantees the quadratic stability of the closed loop uncertain time-delay system as well as an upper bound on the closed loop cost function. The optimal quadratic guaranteed cost controller can be determined by a search over all parameters \( \varepsilon \) and \( \delta = \delta_1, \ldots, \delta_r \) for which the Riccati equation (eqn. 16) has a positive definite solution. The optimal cost is found via
\[ J_{opt} = \inf \left\{ x_0^T P x_0 + \int_{-\tau_r}^{0} x(s)^T \left( \delta_i H_i' H_i + \mu M \right) x(s) ds \right\} \]

over all \( \varepsilon, \delta \) for which the Riccati equation (eqn. 16) has a positive definite stabilising solution.

Hence, to find the optimal solution a search should be made over all possible \( r + 1 \) parameters for which the Riccati equation (eqn. 16) has a positive definite stabilising solution. Corresponding to each solution of the Riccati equation, there exists a guaranteed cost controller. The optimum guaranteed cost controller is the one which makes \( J_{opt} \) minimum.

4 Illustrative example

In this Section we present an example to illustrate the theory developed in preceding Sections. Consider the uncertain time-delay system described by the state equations
\[ \dot{x}_1(t) = x_1(t) - u(t) \]
\[ \dot{x}_2(t) = (-2 + \Delta(t)) x_2(t) + 0.5 x_1(t - 1) + 2 u(t) \]
where \( x_1(t) = e^{at} \) and \( x_2(t) = 0 \) for \( t \in [-1, 0] \). Also, \( \Delta(t) \) is a scalar uncertain parameter subject to the bound \( |\Delta(t)| \leq 1 \) and \( \tau = 1 \). We wish to construct an optimal quadratic guaranteed cost control for this system which minimises the bound on the cost index
\[ J = \int_{0}^{\infty} \left[ x_1(t)^2 + x_2(t)^2 + u(t)^2 \right] dt \]

This uncertain system and the cost function are of the form of eqns. 10 and 11, with
\[ A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \]
\[ Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \end{bmatrix} \]

Also, we factor \( F \) such that \( G = [0 \ 0.5]' \) and \( H = [1 \ 0] \). Theorem 2 implies that the upper bound on the LQ cost function is
\[ J \leq e^2 P_{11} + \left( \frac{e^2 - 1}{2} \right) \delta \]

To find the required optimal quadratic guaranteed cost control, we must find the values of parameters \( \varepsilon > 0 \) and \( \delta > 0 \) which minimise this upper bound which is a function of the stabilising solution to Riccati equation. A plot of the cost bound against \( \varepsilon \) and \( \delta \) is shown in Fig. 1. In this Figure, corresponding to each value of \( \varepsilon \) and \( \delta \) the quadratic guaranteed cost controller is determined via a solution to the Riccati equation (eqn. 16), and the corresponding value of the upper bound on the LQ cost function is plotted. From this
plot we determine the optimum value of ε and δ to be $\varepsilon^* = 1.4$ and $\delta^* = 0.8$. Corresponding to these values of $\varepsilon$ and $\delta$ we obtain the following stabilising solution to the Riccati equation (eqn. 16):

$$P^+ = \begin{bmatrix} 6.3175 & 1.1221 \\ 1.1221 & 0.5654 \end{bmatrix} > 0$$

Fig. 1 Cost bound against $\varepsilon$ and $\delta$

The corresponding value of the cost bound is $J \approx 48.7247$. Also, eqn. 17 gives the corresponding optimal quadratic guaranteed cost control matrix

$$K = \begin{bmatrix} 4.0732 \\ -0.0087 \end{bmatrix}$$

The optimal quadratic guaranteed cost controller guarantees quadratic stability of the uncertain system. It also minimises the upper bound on the cost function for all values of $\|A\| \leq 1$. A plot of the states of the uncertain system with $\Delta = 0.5$ and the above controller is shown in Fig. 2.

5 References


Fig. 2 States against time for $\Delta = 0.5$